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THE METRIC RELATION OF FEUERBACH POINT & ITS APPLICATIONS

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Abstract. *In this article we study the metric relation of Feuerbach point and its applications in proving distance properties related to this point, we also study about Feuerbach Conic and some concurrencies and collinarities concerned about this point.*
Keywords: *Inner Feuerbach Point, Outer Feuerbach Point, Feuerbach Conic.*

1. INTRODUCTION

The famous Feuerbach theorem ([1] and [2]) states that "The nine-point circle of a triangle is tangent internally to the incircle and externally to each of the excircles". Given triangle *ABC*, the inner Feuerbach point F_e is the point of tangency with the incircle and exterior Feuerbach points F_a , F_b, F_c are the points of tangency with three excircles. In this note we give the metric relations of these four points (inner and outer Feuerbach points) which are useful in giving the minimal proof for all the fundamental distance properties of these points. Using this relation we can also investigate some new properties of these points. In the conclusion of the article we discuss about a new proof of Feuerbach conic mentioned in the article [3].

In the articles [4] and [5], S´andor Nagydobai Kiss gave the synthetic proof of distance properties of the Feuerbach point, but in this short note we prove all those distance properties using a simple metric relation of this point which is the consequence of Feuerbach theorem. The metric relation whatever we deal throughout the article is actually not a new one, basing on the works of Feuerbach and Euler we just formulated it to the present form.

We make use of standard notations in triangle geometry (see [6]). Given triangle *ABC*, denote by *a*, *b*, *c* the lengths of the sides *BC*, *CA*, *AB* respectively, *s* the semiperimeter, Δ the area, and *R*, *r*, r_a , r_b , r_c the circumradius, inradius and ex radii respectively. Its classical centers are circumcenter O, incenter I, three excenters I_a, I_b, I_c, centroid G, Nine point center N and orthocenter H.

Working with the distance formula, we also make use of $S_A = \frac{b^2+c^2-a^2}{2} = bc\cos A$,
$S_B = \frac{c^2+a^2-b^2}{2} = ac\cos B$ and $S_B = \frac{c^2+a^2-b^2}{2} = ac\cos B$
it is clear that

$$aS_A + bS_B + cS_C = abc\left(\cos A + \cos B + \cos C\right) = 4\Delta\left(R+r\right).$$

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And also using Euler's formula we have $OI^2 = R^2 - 2Rr$, $OI_a^{\,2} = R^2 + 2Rr_a$, $OI_b^{\,2} = R^2 + 2Rr_b$ and $OI_c^{\,2} = R^2 + 2Rr_c$

2. MAIN RESULTS

Theorem 2.1. If F_e is the inner Feuerbach point, F_a, F_b , F_c are exterior Feuerbach points of non equilateral triangle ABC and let M be a finite point in the plane of the triangle then

(A). $F_eM^2 = \dfrac{R\,IM^2 - 2r\,NM^2}{R-2r} + \dfrac{Rr}{2}$

(B). $F_aM^2 = \dfrac{R\,I_aM^2 + 2r_a\,NM^2}{R+2r_a} - \dfrac{Rr_a}{2}$

(C). $F_bM^2 = \dfrac{R\,I_bM^2 + 2r_b\,NM^2}{R+2r_b} - \dfrac{Rr_b}{2}$

(D). $F_cM^2 = \dfrac{R\,I_cM^2 + 2r_c\,NM^2}{R+2r_c} - \dfrac{Rr_c}{2}$

Proof:

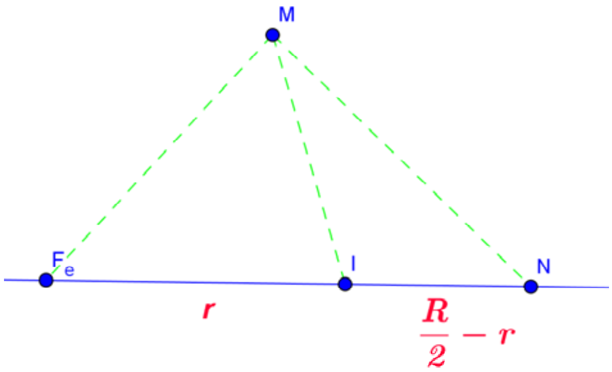

Figure 1.

For proving (A), we apply Stewart's theorem to triangle MF_eN with Cevian IM (provided M doesn't lie on line join of N and F_e) (Fig. 1).

We get,

$$NF_e.IM^2 = F_eI.NM^2 + NI.F_eM^2 - F_eI.IN.NF_e \qquad (1)$$

Using Feuerbach theorem the following metric relations are well known

$$NF_e = NF_a = NF_b = NF_c = \frac{R}{2}$$

$$IF_e = r\,, I_aF_a = r_a\,, I_bF_b = r_b\,,\ I_cF_c = r_c$$

$$IN = \frac{R-2r}{2},\ NI_a = \frac{R+2r_a}{2},\ NI_b = \frac{R+2r_b}{2} \text{ and } NI_c = \frac{R+2r_c}{2}$$

By replacing these metric relations in (1) and further simplification proves (A).

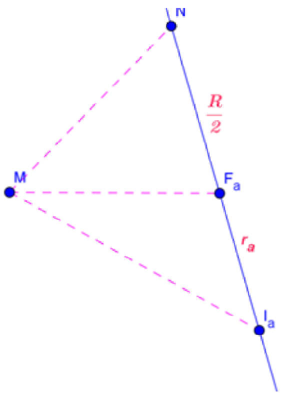

Figure2.

Now for (b), we consider the triangle MNI_a in which MF_a is a Cevian (Fig. 2).
By applying Stewart's theorem to triangle MNI_a, we get

$$NI_a.F_aM^2 = F_aI_a.NM^2 + NF_a.I_aM^2 - NF_a.F_aI_a.NI_a$$

Further simplification by replacing the metric relations obtained from Feuerbach theorem gives (B). Similarly we can prove (C) and (D)

Remark: The metric relations presented in Theorem-2.1 are true even if the point M is collinear with the lines formed by join of (N, F_e), (N, F_a), (N,F_b), (N, F_c) and its proof is quite obvious. In the relations we used the directed line segments only.

Lemma 2.1. If P is any point on the nine point circle then

(a). $F_eP^2 = \frac{R}{R-2r}\left(IP^2 - r^2\right)$ (b). $F_aP^2 = \frac{R}{R+2r_a}\left(I_aP^2 - r_a^2\right)$

(c). $F_bP^2 = \frac{R}{R+2r_b}\left(I_bP^2 - r_b^2\right)$ (d). $F_cP^2 = \frac{R}{R+2r_c}\left(I_cP^2 - r_c^2\right)$

Proof: For proving (a), we proceed as follows:
By fixing M as P in theorem – 2.1(A), we get

$$F_eP^2 = \frac{R\,IP^2 - 2r\,NP^2}{R-2r} + \frac{Rr}{2} \quad (2)$$

Since P is a point on the nine point circle, we have $NP = \frac{R}{2}$. By replacing NP as $\frac{R}{2}$ in (2) and further simplification gives (a). Similarly we can prove (b), (c) and (d).

3. APPLICATIONS OF THEOREM 2.1

Theorem2.2. If $F_e$is the inner Feuerbach point of triangle ABC, and X, Y, Z are the midpoints of the sides BC, CA, AB, respectively, then one of the distances F_e X, F_eY , F_e Z is equal to the sum of other two distances[4, 5, 7].

Proof:

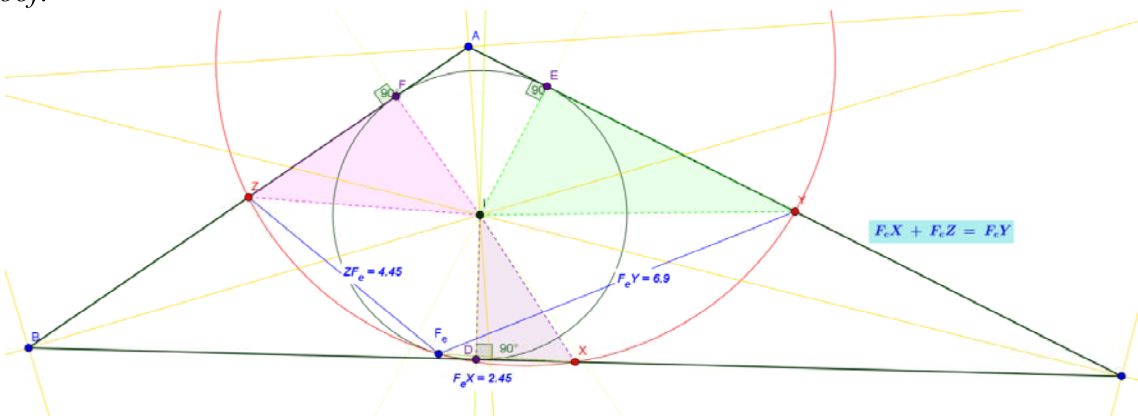

Figure3.

Using Lemma 2.1(a), we have $F_eP^2 = \frac{R}{R-2r}\left(IP^2 - r^2\right)$. Let us fix P as X, Y and Z (since X, Y and Z are the mid points of the sides so they lie on the nine point circle).

We get,

$$F_eX^2 = \frac{R}{R-2r}\left(IX^2 - r^2\right),\ F_eY^2 = \frac{R}{R-2r}\left(IY^2 - r^2\right) \text{ and } F_eZ^2 = \frac{R}{R-2r}\left(IZ^2 - r^2\right)$$

Let D, E, F are the points of contact of incircle with the sides BC, CA and AB (Fig. 3). So BD=BF=s-b, CD=CE=s-c and AE=AF=s-a

Hence from triangle DIX, by Pythagoras theorem $IX^2 = ID^2 + DX^2$

$$\Rightarrow IX^2 - r^2 = DX^2 = \left(BX - BD\right)^2 = \left(CD - CX\right)^2 = \left(s - b - \frac{a}{2}\right)^2 = \left(\frac{a}{2} - s + c\right)^2 = \frac{1}{4}\left(b-c\right)^2$$

So $F_eX^2 = \frac{R}{R-2r}\left(IX^2 - r^2\right) = \frac{R}{R-2r}\left(\frac{b-c}{2}\right)^2 = \frac{R^2}{4OI^2}\left(b-c\right)^2$. It implies

$$F_eX = \frac{R}{2OI}\left|b-c\right|$$

Similarly we can prove $F_eY = \frac{R}{2OI}\left|c-a\right|$ and $F_eZ = \frac{R}{2OI}\left|a-b\right|$. Without loss of generality let us consider $a \ge b \ge c$. It implies one out of the three $\left|a-b\right|$, $\left|b-c\right|$ and $\left|c-a\right|$ is equal to the sum of the remaining two.It follows one of the distances F_eX, F_eY , F_eZ is equal to the sum of other two distances.

Theorem2.3. If the nine-point circle touches the A-excircle at F_a , then one of F_aX, F_aY , F_aZ is the sum of the remaining two [4, 5, 7].

Proof: Using Lemma 2.1(b) , we have

$$F_aP^2 = \frac{R}{R+2r_a}\left(I_aP^2 - r_a^{\,2}\right) \tag{3}$$

Let us fix P as X, since X is also lies on nine point circle, we get

$$F_aX^2 = \frac{R}{R+2r_a}\left(I_aX^2 - r_a^{\,2}\right)$$

Let D_A, E_A, F_A are the points of contact of A-excircle with the sides BC, CA and AB(Fig. 4). So $BD_A = BF_A$=s-c, $AE_A = AF_A$ = s, $CD_A = CE_A$=s-b and $AE_A = AF_A$ = s. Hence from triangle D_AI_aX, by Pythagoras theorem $I_aX^2 = I_aD_A^{\,2} + D_AX^2$

$$\Rightarrow I_aX^2 - r_a^{\,2} = D_AX^2 = \left(BX - BD_A\right)^2 = \left(CD_A - CX\right)^2 = \left(\frac{a}{2} - s + c\right)^2 = \left(s - b - \frac{a}{2}\right)^2 = \frac{1}{4}(b-c)^2$$

So $F_aX^2 = \frac{R}{R+2r_a}\left(I_aX^2 - r_a^{\,2}\right) = \frac{R}{R+2r_a}\left(\frac{b-c}{2}\right)^2 = \frac{R^2}{4OI_a^{\,2}}(b-c)^2$. It implies

$$F_aX = \frac{R}{2OI_a}\left|b-c\right|$$

In the similar manner to get the values F_aY and F_aZ , replace P in (3) with Y and Z (since Y and Z also lie on nine point circle), we get $F_aY^2 = \frac{R}{R+2r_a}\left(I_aY^2 - r_a^{\,2}\right)$ and

$$F_aZ^2 = \frac{R}{R+2r_a}\left(I_aZ^2 - r_a^{\,2}\right)$$

Now it is very clear that for triangles E_AI_aY, F_AI_aZ, using Pythagoras theorem, we have

$$I_aY^2 = I_aE_A^{\,2} + E_AY^2 \text{ and } I_aZ^2 = I_aF_A^{\,2} + F_AZ^2$$

$$\Rightarrow I_aY^2 - r_a^{\,2} = E_AY^2 = \left(CY + CE_A\right)^2 = \left(AE_A - AY\right)^2 = \left(\frac{b}{2} + s - b\right)^2 = \left(s - \frac{b}{2}\right)^2 = \frac{1}{4}(a+c)^2$$

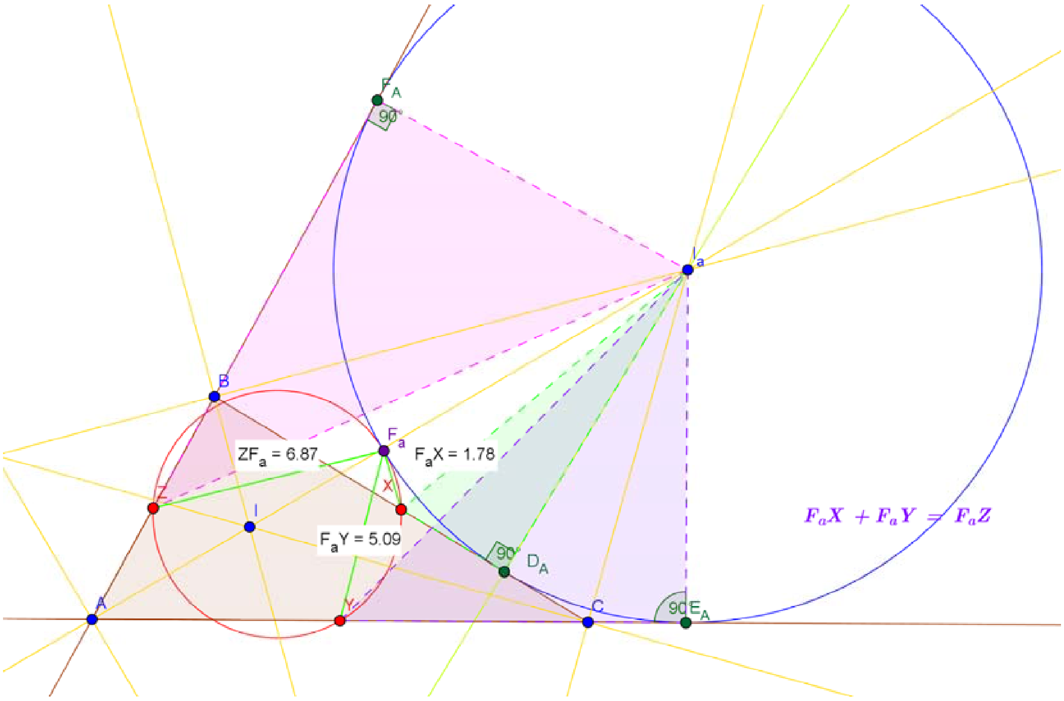

Figure 4.

Similarly

$$\Rightarrow I_aZ^2 - r_a^2 = F_AZ^2 = \left(BZ + BF_A\right)^2 = \left(AF_A - AZ\right)^2 = \left(\frac{c}{2} + s - c\right)^2 = \left(s - \frac{c}{2}\right)^2 = \frac{1}{4}\left(a+b\right)^2$$

So

$$F_aY^2 = \frac{R}{R+2r_a}\left(I_aY^2 - r_a^2\right) = \frac{R}{R+2r_a}\left(\frac{a+b}{2}\right)^2 = \frac{R^2}{4OI_a^2}\left(a+b\right)^2 \Rightarrow F_aY = \frac{R}{2OI_a}\left|a+b\right|$$

and $F_aZ^2 = \frac{R}{R+2r_a}\left(I_aZ^2 - r_a^2\right) = \frac{R}{R+2r_a}\left(\frac{a+c}{2}\right)^2 = \frac{R^2}{4OI_a^2}\left(a+c\right)^2 \Rightarrow F_aZ = \frac{R}{2OI_a}\left|a+c\right|$.

Without loss of generality let us consider $a \geq b \geq c$. It implies one out of the three $|b-c|$, $|a+b|$ and $|a+c|$ is equal to the sum of the remaining two. It follows one of F_aX, F_aY, F_aZ is the sum of the remaining two.

Remark: The similar results mentioned in Theorem 2.3 are also holds true, for the other two excircles. For proof we can proceed as we dealt in Theorem2.3.

Theorem 2.4. The distances from the Feuerbach point $F_e$to the vertices of triangle ABC are given by

$$AF_e^{\,2} = \frac{(s-a)^2 R - S_A}{R-2r},\ BF_e^{\,2} = \frac{(s-b)^2 R - S_B}{R-2r} \text{ and } CF_e^{\,2} = \frac{(s-c)^2 R - S_C}{R-2r} \quad [4, 5, 7]$$

Proof: Using Theorem2.1(A), we have

$$F_eM^2 = \frac{R\,IM^2 - 2r\,NM^2}{R-2r} + \frac{Rr}{2} \tag{2.1A}$$

Since (2.1A) is true for any M let us fix M as A, B and C, we get,

$$\left.\begin{aligned} F_eA^2 &= \frac{R\,IA^2 - 2r\,NA^2}{R-2r} + \frac{Rr}{2} \\ F_eB^2 &= \frac{R\,IB^2 - 2r\,NB^2}{R-2r} + \frac{Rr}{2} \\ F_eC^2 &= \frac{R\,IC^2 - 2r\,NC^2}{R-2r} + \frac{Rr}{2} \end{aligned}\right\} \tag{4}$$

and it is well known that

$$IA^2 = r^2 + (s-a)^2,\ IB^2 = r^2 + (s-b)^2,\ IC^2 = r^2 + (s-c)^2$$

and

$$4NA^2 = b^2 + c^2 - a^2 + R^2 = 2S_A + R^2,\ 4NB^2 = a^2 + c^2 - b^2 + R^2 = 2S_B + R^2,\ 4NC^2 = 2S_C + R^2$$

By replacing these relations in (4) and further simplification proves the conclusions

$$AF_e^{\,2} = \frac{(s-a)^2 R - S_A}{R-2r},\ BF_e^{\,2} = \frac{(s-b)^2 R - S_B}{R-2r} \text{ and } CF_e^{\,2} = \frac{(s-c)^2 R - S_C}{R-2r}$$

Theorem2.5. The distances from the Feuerbach point F_a to the vertices of triangle ABC are given by $AF_a^{\,2} = \frac{s^2R + r_aS_A}{R+2r_a}$, $BF_a^{\,2} = \frac{(s-c)^2 R + r_aS_B}{R+2r_a}$ and $CF_a^{\,2} = \frac{(s-b)^2 R + r_aS_C}{R+2r_a}$ [4, 5, 7].

Proof: Using Theorem2.1(B), we have

$$F_aM^2 = \frac{R\,I_aM^2 + 2r_a\,NM^2}{R+2r_a} - \frac{Rr_a}{2} \tag{2.1B}$$

Since (2.1B) is true for any M, let us fix M as A, B and C, we get

$$\left.\begin{aligned} F_aA^2 &= \frac{R\,I_aA^2 + 2r_a\,NA^2}{R+2r_a} - \frac{Rr_a}{2} \\ F_aB^2 &= \frac{R\,I_aB^2 + 2r_a\,NB^2}{R+2r_a} - \frac{Rr_a}{2} \\ F_aC^2 &= \frac{R\,I_aC^2 + 2r_a\,NC^2}{R+2r_a} - \frac{Rr_a}{2} \end{aligned}\right\} \tag{5}$$

and we are familiar with the results,

$$I_aA^2 = r_a^{\,2} + s^2,\; I_aB^2 = r_a^{\,2} + (s-c)^2,\; I_aC^2 = r_a^{\,2} + (s-b)^2$$

and

$$4NA^2 = 2S_A + R^2,\; 4NB^2 = 2S_B + R^2,\; 4NC^2 = 2S_C + R^2$$

By replacing these relations in (5) and further simplification proves the conclusions

$$AF_a^{\,2} = \frac{s^2R + r_aS_A}{R+2r_a},\; BF_a^{\,2} = \frac{\left(s-c\right)^2 R + r_aS_B}{R+2r_a} \text{ and } CF_a^{\,2} = \frac{\left(s-b\right)^2 R + r_aS_C}{R+2r_a}.$$

Remark: The similar results mentioned in Theorem 2.5 also holds true for the other two excircles. For proof we can proceed as we dealt inTheorem 2.5.

Theorem2.6. If F_e, F_a, F_b and F_c are the inner and outer Feuerbach points then

$$F_eF_a = \frac{|b-c|R^2}{OI.OI_a},\; F_eF_b = \frac{|c-a|R^2}{OI.OI_b} \text{ and } F_eF_c = \frac{|a-b|R^2}{OI.OI_c} \text{ [4, 5, 7].}$$

Proof: For proving first formula we proceed as follows:

Clearly from Lemma 2.1(a), we have, If P is any point on the nine point circle then

$$F_eP^2 = \frac{R}{R-2r}\left(IP^2 - r^2\right) \tag{2.1a}$$

Since (2.1a) is true for any P, let us fix P as F_a , we get

$$F_eF_a^{\,2} = \frac{R}{R-2r}\left(IF_a^{\,2} - r^2\right) \tag{6}$$

but using (2.1B), we have $F_aM^2 = \dfrac{RI_aM^2 + 2r_aNM^2}{R+2r_a} - \dfrac{Rr_a}{2}$

By replacing M as I in (2.1B) we get $F_aI^2 = \frac{RI_aI^2 + 2r_aNI^2}{R+2r_a} - \frac{Rr_a}{2}$. Now from (6),

$$F_eF_a^{\ 2} = \frac{R}{R-2r}\left(\frac{RI_aI^2 + 2r_aNI^2}{R+2r_a} - \frac{Rr_a}{2} - r^2\right) \tag{7}$$

Further simplification by replacing $II_a = 4R\sin\left(\frac{A}{2}\right)$ and $NI = \left(\frac{R-2r}{2}\right)$, gives

$$F_eF_a^{\ 2} = \frac{R}{2(R-2r)(R+2r_a)}\left(32R^3\sin^2\left(\frac{A}{2}\right) + 4r_a\left(\frac{R-2r}{2}\right)^2 - Rr_a(R+2r_a) - 2r^2(R+2r_a)\right)$$

It further implies $F_eF_a^{\ 2} = \frac{2R^2}{2(R-2r)(R+2r_a)}\left(16R^2\sin^2\left(\frac{A}{2}\right) - (r+r_a)^2\right)$. Further simplification by replacing $r + r_a = 4R\sin\left(\frac{A}{2}\right)\cos\left(\frac{B-C}{2}\right)$, gives

$$F_eF_a^{\ 2} = \frac{16R^4\sin^2\left(\frac{A}{2}\right)}{(R-2r)(R+2r_a)}\left(1-\cos^2\left(\frac{B-C}{2}\right)\right)$$

$$\Rightarrow F_eF_a^{\ 2} = \frac{4R^6\,4\sin^2\left(\frac{A}{2}\right)\sin^2\left(\frac{B-C}{2}\right)}{OI^2.OI_a^{\ 2}} = \frac{4R^6}{OI^2.OI_a^{\ 2}}(\sin B - \sin C)^2 = \frac{R^4}{OI^2.OI_a^{\ 2}}(b-c)^2$$

It implies $F_eF_a = \frac{R^2}{OI.OI_a}|b-c|$. Similarly we can prove $F_eF_b = \frac{|c-a|R^2}{OI.OI_b}$ and $F_eF_c = \frac{|a-b|R^2}{OI.OI_c}$.

Theorem2.7. If F_a, F_b and F_c are outer Feuerbach points then

$$F_aF_b = \frac{(a+b)R^2}{OI_a.OI_b},\ F_bF_c = \frac{(b+c)R^2}{OI_b.OI_c} \text{ and } F_cF_a = \frac{(c+a)R^2}{OI_c.OI_a} \text{ [4, 5, 7].}$$

Proof: For proving first formula we proceed as follows:

Clearly from Lemma 2.1(b), we have, If P is any point on the nine point circle then

$$F_aP^2 = \frac{R}{R+2r_a}\left(I_aP^2 - r_a^{\ 2}\right) \tag{2.1b}$$

Now since (2.1b) is true for any P, let us fix P as F_b , we get

$$F_aF_b^{\,2} = \frac{R}{R+2r_a}\left(I_aF_b^{\,2} - r_a^{\,2}\right) \qquad (8)$$

But using (2.1C), we have $F_bM^2 = \dfrac{RI_bM^2 + 2r_bNM^2}{R+2r_b} - \dfrac{Rr_b}{2}$ (2.1C)

by replacing M as I_a in (2.1C) we get $F_bI_a^{\,2} = \dfrac{RI_bI_a^{\,2} + 2r_bNI_a^{\,2}}{R+2r_b} - \dfrac{Rr_b}{2}$

Now from (7),

$$F_aF_b^{\,2} = \frac{R}{R+2r_a}\left(\frac{RI_aI_b^{\,2} + 2r_bNI_a^{\,2}}{R+2r_b} - \frac{Rr_b}{2} - r_a^{\,2}\right) \qquad (9)$$

Further simplification by replacing $I_aI_b = 4R\cos\left(\dfrac{C}{2}\right)$ and $NI_a = \left(\dfrac{R+2r_a}{2}\right)$, gives

$$F_aF_b^{\,2} = \frac{R}{2\left(R+2r_a\right)\left(R+2r_b\right)}\left(32R^3\cos^2\left(\frac{C}{2}\right) + 4r_b\left(\frac{R+2r_a}{2}\right)^2 - Rr_b\left(R+2r_b\right) - 2r_a^{\,2}\left(R+2r_b\right)\right)$$

It further implies $F_aF_b^{\,2} = \dfrac{2R^2}{2\left(R+2r_a\right)\left(R+2r_b\right)}\left(16R^2\cos^2\left(\dfrac{C}{2}\right) - \left(r_a - r_b\right)^2\right)$.

Further simplification by replacing $r_a - r_b = 4R\cos\left(\dfrac{C}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$ gives

$$F_aF_b^{\,2} = \frac{16R^4\cos^2\left(\frac{C}{2}\right)}{\left(R+2r_a\right)\left(R+2r_b\right)}\left(1 - \sin^2\left(\frac{A-B}{2}\right)\right) \Rightarrow F_aF_b^{\,2} = \frac{4R^6\,4\cos^2\left(\frac{C}{2}\right)\cos^2\left(\frac{A-B}{2}\right)}{OI_a^{\,2}.OI_b^{\,2}}$$

$$= \frac{4R^6}{OI_a^{\,2}.OI_b^{\,2}}\left(\sin A + \sin B\right)^2 = \frac{R^4}{OI_a^{\,2}.OI_b^{\,2}}\left(a+b\right)^2$$

It implies $F_aF_b = \dfrac{R^2}{OI_a.OI_b}\left(a+b\right)$. Similarly we can prove

$$F_bF_c = \frac{\left(b+c\right)R^2}{OI_b.OI_c} \text{ and } F_cF_a = \frac{\left(c+a\right)R^2}{OI_c.OI_a}$$

Theorem2.8. If F_e, F_a, F_b and F_c are the inner and outer Feuerbach points then

$$\frac{\left(F_aI^2 - r^2\right)}{\left(F_eI_a^{\,2} - r_a^{\,2}\right)}\left(R+2r_a\right) = \frac{\left(F_bI^2 - r^2\right)}{\left(F_eI_b^{\,2} - r_b^{\,2}\right)}\left(R+2r_b\right) = \frac{\left(F_cI^2 - r^2\right)}{\left(F_eI_c^{\,2} - r_c^{\,2}\right)}\left(R+2r_c\right) = \left(R-2r\right)$$

Proof: It is clear from Lemma 2.1(a), $F_eP^2 = \frac{R}{R-2r}\left(IP^2 - r^2\right)$.

Fix P as F_a , we get

$$F_eF_a^{\,2} = \frac{R}{R-2r}\left(IF_a^{\,2} - r^2\right) \quad (10)$$

Similarly by fixing P as F_e in Lemma 2.1(b), we get

$$F_aF_e^{\,2} = \frac{R}{R+2r_a}\left(I_aF_e^{\,2} - r_a^{\,2}\right) \quad (11)$$

Now from (10) and (11), it is clear that

$$\frac{F_aI^2 - r^2}{F_eI_a^{\,2} - r_a^{\,2}} = \frac{R-2r}{R+2r_a} = \frac{NI}{NI_a} = \frac{OI}{OI_a} \quad (12)$$

Similarly we can prove

$$\frac{F_bI^2 - r^2}{F_eI_b^{\,2} - r_b^{\,2}} = \frac{R-2r}{R+2r_b} = \frac{NI}{NI_b} = \frac{OI}{OI_b} \quad (13)$$

and $\frac{F_cI^2 - r^2}{F_eI_c^{\,2} - r_c^{\,2}} = \frac{R-2r}{R+2r_c} = \frac{NI}{NI_c} = \frac{OI}{OI_c}$ (14)

Using (12), (13) and (14) we can prove the desired result

$$\frac{\left(F_aI^2 - r^2\right)}{\left(F_eI_a^{\,2} - r_a^{\,2}\right)}\left(R+2r_a\right) = \frac{\left(F_bI^2 - r^2\right)}{\left(F_eI_b^{\,2} - r_b^{\,2}\right)}\left(R+2r_b\right) = \frac{\left(F_cI^2 - r^2\right)}{\left(F_eI_c^{\,2} - r_c^{\,2}\right)}\left(R+2r_c\right) = \left(R-2r\right)$$

Theorem2.9. If F_a, F_b and F_c are outer Feuerbach points then

$$\left(F_aI_b^{\,2} - r_b^{\,2}\right)\left(F_bI_c^{\,2} - r_c^{\,2}\right)\left(F_cI_a^{\,2} - r_a^{\,2}\right) = \left(F_bI_a^{\,2} - r_a^{\,2}\right)\left(F_cI_b^{\,2} - r_b^{\,2}\right)\left(F_aI_c^{\,2} - r_c^{\,2}\right)$$

Proof: It is clear from lemma-2.1(b), $F_aP^2 = \frac{R}{R+2r_a}\left(I_aP^2 - r_a^{\,2}\right)$

Fix P as F_b , we get

$$F_aF_b^{\,2} = \frac{R}{R+2r_a}\left(I_aF_b^{\,2} - r_a^{\,2}\right) \quad (15)$$

Similarly by fixing P as $F_a$in lemma-2.1(c), we get

$$F_bF_a^{\,2} = \frac{R}{R+2r_b}\left(I_bF_a^{\,2} - r_b^{\,2}\right) \quad (16)$$

Now from (15) and (16), it is clear that

$$\frac{F_aI_b^{\,2}-r_b^{\,2}}{F_bI_a^{\,2}-r_a^{\,2}}=\frac{R+2r_b}{R+2r_a}=\frac{NI_b}{NI_a}=\frac{OI_b}{OI_a} \tag{17}$$

Similarly we can prove

$$\frac{F_bI_c^{\,2}-r_c^{\,2}}{F_cI_b^{\,2}-r_b^{\,2}}=\frac{R+2r_c}{R+2r_b}=\frac{NI_c}{NI_b}=\frac{OI_c}{OI_b} \tag{18}$$

and

$$\frac{F_cI_a^{\,2}-r_a^{\,2}}{F_aI_c^{\,2}-r_c^{\,2}}=\frac{R+2r_a}{R+2r_c}=\frac{NI_a}{NI_c}=\frac{OI_a}{OI_c} \tag{19}$$

By multiplying (17), (18) and (19) we get desired result

$$\left(F_aI_b^{\,2}-r_b^{\,2}\right)\left(F_bI_c^{\,2}-r_c^{\,2}\right)\left(F_cI_a^{\,2}-r_a^{\,2}\right)=\left(F_bI_a^{\,2}-r_a^{\,2}\right)\left(F_cI_b^{\,2}-r_b^{\,2}\right)\left(F_aI_c^{\,2}-r_c^{\,2}\right)$$

To discuss one more theorem related to these points (inner and outer Feuerbach points) we need a list of lemmas which we are presenting here without proofs.

Lemma 2.2. Let A_a, A_b, A_c are the feet of internal and external angular bisector of angle A(A_b closer to the vertex B and A_c closer to the vertex C) on the side BC respectively, similarly define the points B_a, B_b, B_c, C_a, C_b and C_c. . If F_e, F_a, F_b and F_c are the inner and outer Feuerbach points then the following pair of triangles are directly similar [8, 9].

(a) The triangle with feet of internal angular bisectors of the angles A, B and C as vertices is directly similar to the triangle formed by outer Feuerbach points.
That is $\Delta A_aB_bC_c \sim \Delta F_aF_bF_c$.

(b) The triangle with feet of two external angular bisectors and one internal angular bisector as vertices is directly similar to the triangle formed by two outer Feuerbach points and one inner Feuerbach point.
That is $\Delta A_bB_aC_c \sim \Delta F_bF_aF_e$, $\Delta A_cB_bC_a \sim \Delta F_cF_eF_a$ and $\Delta A_aB_cC_b \sim \Delta F_eF_cF_b$.

Lemma 2.3. The lines AF_a, BF_b and CF_c are concurrent at X(12), which is collinear with the points F_e , I and N. [10]

Lemma 2.4. The following set of points is collinear [8]:
(a) The points F_e, A_a, F_a are collinear.
(b) The points F_e, B_b, F_b are collinear.
(c) The points F_e, C_c, F_c are collinear.
(d) The points either C_b, F_a, F_b, or C_a, F_a, F_b are collinear.
(e) The points either B_c, F_a, F_c, or B_a, F_a, F_c are collinear.
(f) The points either A_c, F_b, F_c, or A_b , F_b, F_c,are collinear.

That is the feet of internal and external bisectors lies in 6 lines defined by F_e, F_a, F_b, F_c [11-13].

Lemma 2.5. Let A_a , B_b and C_c be the feet of internal bisectors on respective sides of triangle. Lines $A_a B_b = l_1$, $B_b C_c = l_2$, $C_c A_a = l_3$ are called the axes of internal bisectors. Then each of these lines passes through the foot of the respective external bisector and that l_k is perpendicular to line IO_k ($k = 1, 2, 3$) [11].

The feet of external bisectors are collinear. (This line is called the axe of external bisectors, we note it l). Line l is perpendicular to line IO [11].

Lemma 2.6. Let ${}_bF_a$, ${}_cF_a$ are the points of intersection of the lines F_aF_c and F_aF_b with side BC, similiarly the points ${}_aF_b$, ${}_cF_b$, ${}_aF_c$, ${}_bF_c$ are defined then

(a) The lines through the points A, ${}_cF_a$ and B, ${}_cF_b$ are concurrent and the point of concurrency lies on the internal angular bisector (CC_c) of angle C.

(b) The lines through the points C, ${}_aF_c$ and B, ${}_aF_b$ are concurrent and the point of concurrency lies on the Internal angular bisector (AA_a) of angle A.

(c) The lines through the points A, ${}_bF_a$ and C, ${}_bF_c$ are concurrent and the point of concurrency lies on the internal angular bisector (BB_b) of angle A.

For (a), (b), (c) see Fig. 5.

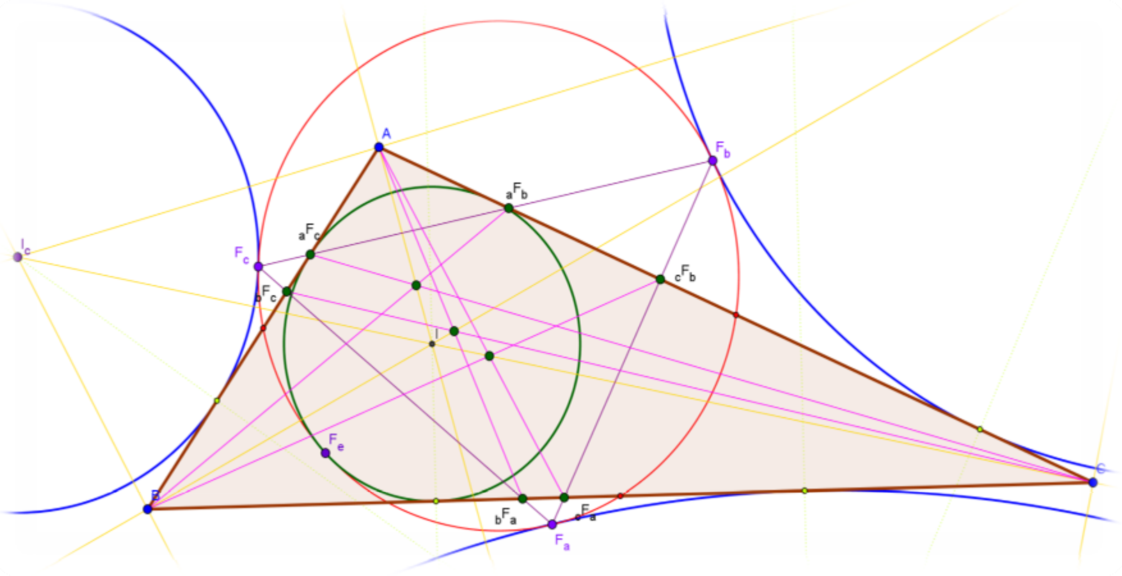

Figure 5.

(d) The lines through the points A, ${}_bF_a$ and B, ${}_aF_b$ are concurrent and the point of concurrency let us call as V_C .

(e) The lines through the points C, ${}_bF_c$ and B, ${}_cF_b$ are concurrent and the point of concurrency let us call as V_A .

(f) The lines through the points A, ${}_cF_a$ and C, ${}_aF_c$ are concurrent and the point of concurrency let us call as V_B.

(g) The points V_A, V_B and V_C are collinear.

(h) The lines formed by join of (C, V_c), (B, V_b)and (A, V_a) are concurrent at V.

For (d), (e), (f), (g), (h) see Fig. 6.

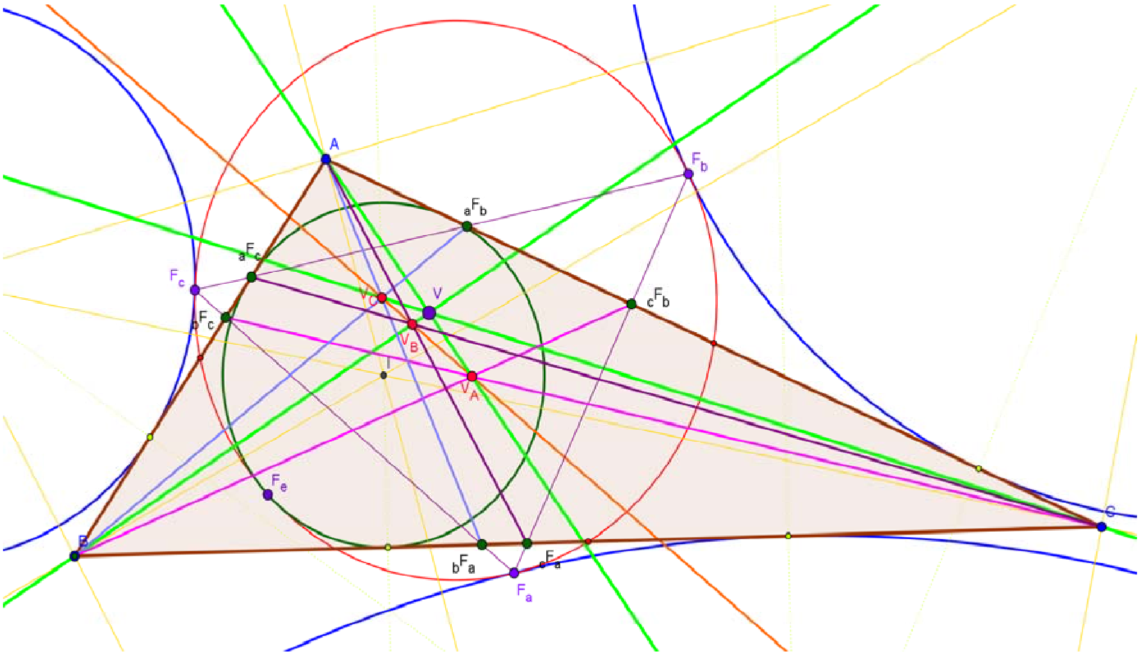

Figure 6.

Theorem 2.10. The six points ${}_bF_a$, ${}_cF_a$, ${}_cF_b$, ${}_aF_b$, ${}_aF_c$ and ${}_bF_c$ are defined as stated in lemma-2.5, then these six points lie on a conic , for the recognition sake let us call this conic as Feuerbach conic [3].

Proof: Consider the hexagon whose vertices are ${}_bF_a$, ${}_cF_a$, ${}_cF_b$, ${}_aF_b$, ${}_aF_c$ and ${}_bF_c$. clearly these six points lies on lines F_aF_b, F_bF_c and F_cF_a in the particular order.

Let us fix $L = {}_bF_a\,{}_cF_a \cap {}_aF_b\,{}_aF_c$, $M = {}_cF_a\,{}_cF_b \cap {}_aF_c\,{}_bF_c$ and $N = {}_cF_b\,{}_aF_b \cap {}_bF_c\,{}_bF_a$.

So as to prove the six points lie on a conic, it is enough to prove that using the converse of Pascal theorem [14], the points L, M and N are collinear (Fig. 7).

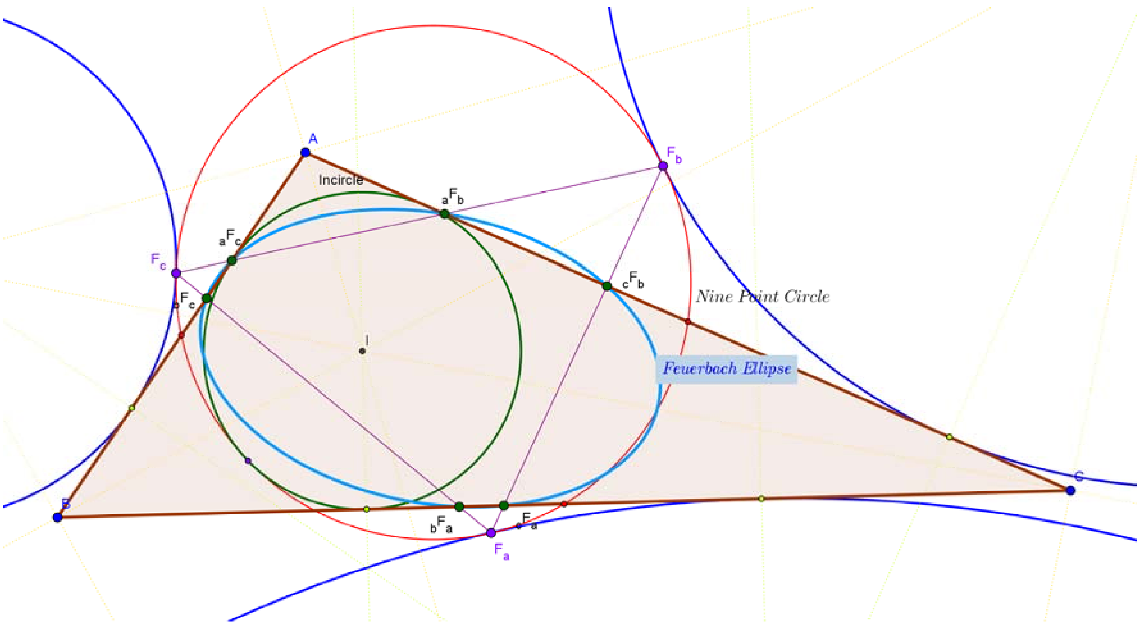

Figure 7.

Clearly the line through ${}_bF_a$, ${}_cF_a$ is BC and the line through ${}_aF_b$, ${}_aF_c$ is F_bF_c. Hence their point of intersection L is the feet of external angular bisector (using Lemma2.5). Similarly M and N are the feet of external angular bisectors.That is L= A_b or A_c , M = C_a or C_b and N= B_a or B_c. So by Lemma2.6, the points L, M and N are lie on the axe of external bisectors.Hence they are collinear and the line through L, M and N acts as Pascal line (axe of external bisector) (Fig. 8).

It proves that the six points lie on a conic (*Feuerbach Conic*).

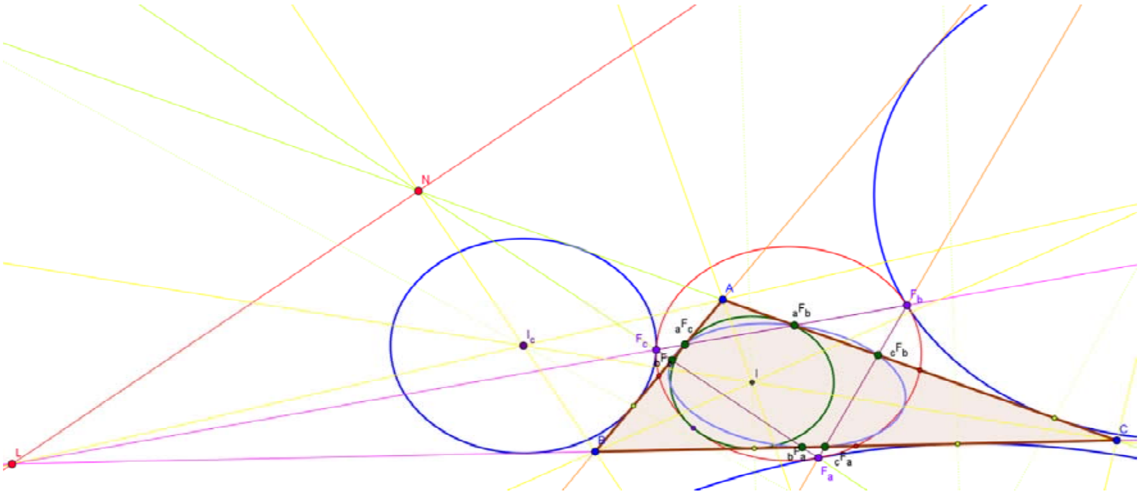

Figure 8.

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