ORIGINAL PAPER

CONJUGATE LAPLACIAN MATRIX OF A GRAPH

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Abstract. *Let G be a simple graph of order n. Let* $c = a + b\sqrt{m}$ *and* $\bar{c} = a - b\sqrt{m}$*, where a and b are two nonzero integers and m is a positive integer such that m is not a perfect square. Let* $P_{L(G)}(\lambda) = |\lambda I - L(G)|$ *is denote the Laplacian characteristic polynomial of a graph G. In this study we define that* $L^c = [l_{ij}^c]$ *is the conjugate Laplacian matrix of the graph G and define that* $P_{L(G)}^c(\lambda) = |\lambda I - L^c(G)|$ *is called the conjugate Laplacian characteristic polynomial of a graph G and study their properties.*
Keywords: *Graph, the Laplacian characteristic polynomial, the conjugate Laplacian characteristic polynomial, eigenvalue*

1. INTRODUCTION

Throughout this paper, we consider only simple connected graphs, i.e. connected graphs without loops and multiple edges. For a graph G, V(G), E(G) denote the set of all vertices and edges, respectively. For a graph G, the degree of a vertex v is the number of edges incident to v and denoted by $deg_v(G)$ or $d(v)$.

The spectrum of a simple graph G of order n consists of the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of its (0,1) adjacency matrix A=A(G) and is denoted by $\sigma(G)$. The Seidel spectrum of G consists of the eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of its (0,-1,1) adjacency matrix $A^* = A^*(G)$ and is denoted by $\sigma^*(G)$. Let $P_G(\lambda) = |\lambda I - A|$ and $P_G^*(\lambda) = |\lambda I - A^*|$ denote the characteristic polynomial and the Seidel characteristic polynomial, respectively [4].

Let $c = a + b\sqrt{m}$, where a and b are two nonzero integers and m is a positive integer such that m is not a perfect square. The value $\bar{c} = a - b\sqrt{m}$ is called the conjugate number of c. We say that $A^c = [c_{ij}]$ is the conjugate adjacency matrix of G if $c_{ij} = c$ for any two adjacent vertices i and j, $c_{ij} = \bar{c}$ for any two nonadjacent vertices i and j, and $c_{ij} = 0$ if i=j. The conjugate spectrum of G is the set of the eigenvalues $\lambda_1^c \geq \lambda_2^c \geq \cdots \geq \lambda_n^c$ of its conjugate adjaceny matrix $A^c = A^c(G)$ and is denoted by $\sigma^c(G)$. Let $P_G^c(\lambda) = |\lambda I - A^c|$ denote the conjugate characteristic polynomial of G. We say that two graphs G and H are conjugate cospectral if $P_G^c(\lambda) = P_H^c(\lambda)$ [4].

In this paper, we define the conjugate Laplacian matrix, the Laplacian Seidel matrix and study their eigenvalues.

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2. MAIN RESULTS

Definition 1.
The conjugate Laplacian matrix of a graph G is $L^c(G) = (l_{ij}^c)_{nxn}$, where

$$l_{ij}^c = \begin{cases} d_i c + (n - d_i - 1)\bar{c} & ; \quad i = j \\ -c & ; \quad i \sim j \\ -\bar{c} & ; \quad i \nsim j. \end{cases}$$

Definition 2.1
The conjugate Laplacian spectrum of G is called to set that consists of the eigenvalues $\lambda_1^c \geq \lambda_2^c \geq \cdots \geq \lambda_n^c$ of the conjugate Laplacian matrix $L^c = L^c(G)$ and is denoted by $\sigma^c(L^c(G))$.

Definition 2.
The Laplacian Seidel matrix of a graph G is $L^*(G) = (l_{ij}^*)_{nxn}$, where

$$l_{ij}^* = \begin{cases} -2d_i & ; \quad i = j \\ 3 & ; \quad i \sim j \\ 1 & ; \quad i \nsim j. \end{cases}$$

Definition 2.2
The Laplacian Seidel spectrum of G is called to set that consists of the eigenvalues $\lambda_1^* \geq \lambda_2^* \geq \cdots \geq \lambda_n^*$ of the Laplacian Seidel matrix $L^* = L^*(G)$ and is denoted by $\sigma^*\big(L^*(G)\big)$.

Theorem 1. [4]
Let A and B be two real symmetric matrices of order n and let $C = A + B$. Then

(1^0) $\lambda_{i+j+1}(C) \leq \lambda_{i+1}(A) + \lambda_{j+1}(B)$,

(2^0) $\lambda_{n-i-j}(C) \geq \lambda_{n-i}(A) + \lambda_{n-j}(B)$,

where $0 \leq i, j, i + j + 1 \leq n$ *[1]*.

Definition 3.
Let G and H be two graphs of order n. Then G and H are the conjugate Laplacian cospectral if $P_{L(G)}^c(\lambda) = P_{L(H)}^c(\lambda)$.

Definition 4.
Let G and H be two graphs of order n. Then $P_{L(G)}^c(\lambda) = P_{L(H)}^c(\lambda)$ if and only if $P_{L(G)}(\lambda) = P_{L(H)}(\lambda)$ and $P_{L(\bar{G})}(\lambda) = P_{L(\bar{H})}(\lambda)$.

Theorem 2.
If G and H are two the conjugate Laplacian cospectral graphs the their complementary graphs $\bar{G}$ ve $\bar{H}$ are also the conjugate Laplacian cospectral.

Proof: Let $P^c_{L(G)}(\lambda)=\sum_{k=0}^{n}(a_k+b_k\sqrt{m})\lambda^{n-k}$ and $P^c_{L(H)}(\lambda)=\sum_{k=0}^{n}(c_k+d_k\sqrt{m})\lambda^{n-k}$. Since G and H two the conjugate Laplacian cospectral we have $P^c_{L(G)}(\lambda)=P^c_{L(H)}(\lambda)$. Then we have

$$(a_k-c_k)+(b_k-d_k)\sqrt{m}=0.$$

Since $\sqrt{m}$ is an irrational number it turns out that $a_k=c_k$ and $b_k=d_k$ for k=0,1,…,n. Since $P^c_{L(\bar{G})}(\lambda)=\sum_{k=0}^{n}(a_k-b_k\sqrt{m})\lambda^{n-k}$, we obtain that $P^c_{L(\bar{G})}(\lambda)=P^c_{L(\bar{H})}(\lambda)$.

Theorem 3.
If G is a graph of order $n>2$ then the conjugate Laplacian characteristic polynomial of G is determined uniquely by the collection $\mathcal{P}^c(G)$ of conjugate Laplacian characteristic polynomials of vertex-deleted subgraphs of the graph G.

Proof: Let G and H be two graphs of order $n>2$ such that $\mathcal{P}^c(G)=\mathcal{P}^c(H)$. Since $P^c_{G_i}(\lambda)=P^c_{H_i}(\lambda)$ we obtain from definition 4 that if $P_{L(G_i)}(\lambda)=P_{L(H_i)}(\lambda)$ and $P_{L(\overline{G_i})}(\lambda)=P_{L(\overline{H_i})}(\lambda)$ for $i=1,2,\dots,n$. Consequently, according to [2] it turns out that $P_{L(G)}(\lambda)=P_{L(H)}(\lambda)$ and $P_{L(\bar{G})}(\lambda)=P_{L(\bar{H})}(\lambda)$, which proves that $P^c_{L(G)}(\lambda)=P^c_{L(H)}(\lambda)$.
Next, replacing λ with $x+y\sqrt{m}$ note that the conjugate Laplacian characteristic polynomial $P^c_{L(G)}(\lambda)$ can be transformed into the form

$$P^c_G\left(x+y\sqrt{m}\right)=Q_n(x,y)+\sqrt{m}R_n(x,y) \quad (1)$$

where $Q_n(x,y)$ and $R_n(x,y)$ are two polynomials of order n in variables x and y, whose coefficients are integers. Since $P^c_{\bar{G}}\left(x-y\sqrt{m}\right)=\overline{P^c_G\left(x+y\sqrt{m}\right)}$ [4], we have

$$P^c_{\bar{G}}\left(x-y\sqrt{m}\right)=Q_n(x,y)-\sqrt{m}R_n(x,y). \quad (2)$$

We note from (5) and (6) that $x_0+y_0\sqrt{m}\in\sigma^c(G)$ and $x_0-y_0\sqrt{m}\in\sigma^c(\bar{G})$ if and only if x_0 and y_0 is a solution of the following system of equations:

$$Q_n(x,y)=0 \text{ and } R_n(x,y)=0. \quad (3)$$

Proposition 1.
Let $\lambda_1^c\ge\lambda_2^c\ge\cdots\ge\lambda_n^c$ and $\bar{\lambda}_1^c\ge\bar{\lambda}_2^c\ge\cdots\ge\bar{\lambda}_n^c$ be the conjugate Laplacian eigenvalues of the graphs G and $\bar{G}$, respectively. Then $x_{ij}=(\lambda_i^c+\bar{\lambda}_j^c)/2$ and $y_{ij}=(\lambda_i^c-\bar{\lambda}_j^c)/2\sqrt{m}$ is a solution of (3) for any $i,j=1,2,\dots,n$.

Proof: Since $P^c_{L(G)}(\lambda_i^c)=P^c_{L(G)}(x_{ij}+y_{ij}\sqrt{m})$ and $P^c_{L(\bar{G})}\left(\bar{\lambda}_j^c\right)=P^c_{L(\bar{G})}\left(x_{ij}-y_{ij}\sqrt{m}\right)$, making use of (1) and (2) we find that $Q_n\left(x_{ij},y_{ij}\right)=0$ and $R_n\left(x_{ij},y_{ij}\right)=0$.

Proposition 2.
Let $P^c_G\left(x+y\sqrt{m}\right)=Q_n(x,y)+\sqrt{m}R_n(x,y)$. Then the following two equalities are satisfied:

$$\frac{\partial Q_n(x,y)}{\partial x}=\frac{\partial R_n(x,y)}{\partial y} \text{ and } \frac{\partial Q_n(x,y)}{\partial y}=m\frac{\partial R_n(x,y)}{\partial x}.$$

Proof: Let $=x+y\sqrt{m}$. Using the rule for the differentiation of compound functions, we have

$\frac{d\,P^{c}_{L(G)}(\lambda)}{d\lambda}\frac{\partial(x+y\sqrt{m})}{\partial x}=\frac{\partial Q_n(x,y)}{\partial x}+\sqrt{m}\frac{\partial R_n(x,y)}{\partial x}$,
$\frac{d\,P^{c}_{L(G)}(\lambda)}{d\lambda}\frac{\partial(x+y\sqrt{m})}{\partial y}=\frac{\partial Q_n(x,y)}{\partial y}+\sqrt{m}\frac{\partial R_n(x,y)}{\partial y}$,
wherefrom we get $(\frac{\partial Q_n(x,y)}{\partial x}-\frac{\partial R_n(x,y)}{\partial y})\sqrt{m}$-$(\frac{\partial Q_n(x,y)}{\partial y}-m\frac{\partial R_n(x,y)}{\partial x})$ =0. Since $\sqrt{m}$ is an irrational number and keeping in mind that the coefficients of the polynomials $Q_n(x,y)$ and $R_n(x,y)$ are integers, we obtain the statement.

Proposition 3.
For any graph G of order n we have

$$\lambda_{i+1}\left(\frac{1}{a-b\sqrt{m}}L^{c}\right)+\lambda_{n+1-i}\left(-\frac{2b\sqrt{m}}{a-b\sqrt{m}}L\right)+1\leq 0$$

for $i=1,2,\dots,n-1$.

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