

SURFACE FAMILY WITH A COMMON ASYMPTOTIC CURVE IN MINKOWSKI 3-SPACE

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Manuscript received: 22.10.2017; Accepted paper: 15.03.2018;

Published online: 30.06.2018.

Abstract. *In this paper, we express surfaces parametrically through a given spacelike (timelike) asymptotic curve using the Frenet frame of the curve in Minkowski 3-space. Necessary and sufficient conditions for the coefficients of the Frenet frame to satisfy both parametric and asymptotic requirements are derived. We also present some interesting examples to show the validity of this study.*

AMS: 53C40, 53C50.

Keywords: Minkowski 3-space, surface family, asymptotic curve

1. INTRODUCTION

We encounter curves and surfaces in every differential geometry book. Regardless of the representation of the surface, most existing work is focused on the problem: given a surface find and classify special curves, such as geodesic, line of curvatures, asymptotic curves etc., on the surface in question. However, it's very interesting to construct surfaces upon a given curve so that this curve is a special one on the surface.

The concept of family of surfaces having a given characteristic curve was first introduced by Wang et al. [1] in Euclidean 3-space. They proposed a method which produces a surface family with a common geodesic. Kasap et al. [2] introduced new types of marching-scale functions and generalized the work of Wang. In [3] Kasap and Akyıldız defined surfaces with a common geodesic in Minkowski 3-space and gave the sufficient conditions on marching-scale functions so that the given curve is a common geodesic on that surfaces. Şaffak and Kasap [4] constructed surfaces with a common null geodesic.

With the inspiration of work of Wang et al. in [5] authors changed the characteristic curve from geodesic to line of curvature and defined the surface pencil with a common line of curvature. Recently, in [6] Bayram et al. defined the surface pencil with a common asymptotic curve and Bayram and Bilici [7] constructed surfaces with a common involute asymptotic curve. Also, there are lots of studies in different spaces related with constructing surface family with a common special curve [8-20].

In the present paper, we study the problem: given a 3D spacelike (timelike) curve, how we can obtain those surfaces that possess this curve as a common parametric and asymptotic curve in Minkowski 3-space. In section 2, we give some preliminary information about curves and surfaces in Minkowski 3-space and define isoasymptotic curve. We express spacelike surfaces as a linear combination of the Frenet frame of the given curve and derive necessary and sufficient conditions on marching-scale functions to satisfy both parametric and asymptotic requirements in Section 3.

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Section 4 is devoted to timelike surfaces. We illustrate the method by giving some examples. Also, all minimal timelike surfaces are given as examples of timelike surfaces with common asymptotic curve.

2. PRELIMINARIES

Minkowski 3-space is denoted by $\mathbf{R}_1^3 = [\mathbf{R}^3, (+, +, -)]$ and the Lorentzian inner product of $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3) \in \mathbf{R}_1^3$ is defined to be $\langle X, Y \rangle = x_1y_1 + x_2y_2 - x_3y_3$.

A vector $X \in \mathbf{R}_1^3$ is called a spacelike vector when $\langle X, X \rangle > 0$ or $X = 0$. It is called timelike and null (light-like) vector in case of $\langle X, X \rangle < 0$, and $\langle X, X \rangle = 0$ for $X \neq 0$, respectively, [21].

The vector product of vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ in \mathbf{R}_1^3 is defined by [22]

$$X \times Y = (x_3y_2 - x_2y_3, x_1y_3 - x_3y_1, x_1y_2 - x_2y_1).$$

Let $\alpha = \alpha(s)$ be a unit speed curve in \mathbf{R}_1^3 . We denote the natural curvature and torsion of $\alpha(s)$ with $\kappa(s)$ and $\tau(s)$, respectively. Consider the Frenet frame $\{T, N, B\}$ associated with curve $\alpha(s)$ such that $T = T(s)$, $N = N(s)$ and $B = B(s)$ are the unit tangent, the principal normal and the binormal vector fields, respectively. If $\alpha = \alpha(s)$ is a spacelike curve, then the Frenet formulas are given as

$$T'(s) = \kappa(s)N(s), \quad N'(s) = \varepsilon\kappa(s)T(s) + \tau(s)B, \quad B'(s) = \tau(s)N(s),$$

where

$$\varepsilon = \begin{cases} -1, & B \text{ is timelike,} \\ 1, & B \text{ is spacelike.} \end{cases}$$

If $\alpha = \alpha(s)$ is a timelike curve, then above equations are given as [23]

$$T'(s) = \kappa(s)N(s), \quad N'(s) = \kappa(s)T(s) - \tau(s)B(s), \quad B'(s) = \tau(s)N(s).$$

A surface in \mathbf{R}_1^3 is called a timelike surface if the induced metric on the surface is a Lorentz metric and is called a spacelike surface if the induced metric on the surface is a positive definite Riemannian metric, i.e. the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [24].

A surface curve is called an asymptotic curve provided its velocity always points in an asymptotic direction, which is the direction in which the normal curvature is zero [25].

According to the above definition the curve is an asymptotic curve on the surface $P(s, t)$ if and only if

$$\left\langle \frac{\partial n(s, t_0)}{\partial s}, T(s) \right\rangle = 0, \quad (1)$$

where $n(s, t_0)$ is the normal vector of $P(s, t)$ along the curve α .

An isoparametric curve $\alpha = \alpha(s)$ is a curve on a surface $P(s, t)$ in \mathbb{R}_1^3 that has a constant s or t - parameter value, namely, there exists a parameter s_0 or t_0 such that $\alpha(s) = P(s, t_0)$ or $\alpha(t) = P(s_0, t)$. Given a parametric curve $\alpha(s)$, we call it an *isoasymptotic* of the surface P if it is both an asymptotic curve and a parameter curve on P .

We assume that $\kappa(s) \neq 0$ for $\alpha(s)$ along the paper. Otherwise, the principal normal of the curve is undefined or the curve is a straightline.

3. SPACELIKE SURFACES WITH A COMMON SPACELIKE ASYMPTOTIC

Let $P(s, t)$ be a parametric spacelike surface. The surface is defined by a given curve $\alpha = \alpha(s)$ as follows:

$$\begin{cases} P(s, t) = \alpha(s) + u(s, t)T(s) + v(s, t)N(s) + w(s, t)B(s), \\ L_1 \leq s \leq L_2, T_1 \leq t \leq T_2, \end{cases} \quad (2)$$

where $u(s, t)$, $v(s, t)$ and $w(s, t)$ are C^1 functions.

Our aim is to find the necessary and sufficient conditions for which the curve is a parameter curve and an asymptotic curve on the surface.

Firstly, since $\alpha(s)$ is a parameter curve on the surface $P(s, t)$, there exists a parameter $t_0 \in [T_1, T_2]$ such that

$$u(s, t_0) = v(s, t_0) = w(s, t_0) \equiv 0, \quad L_1 \leq s \leq L_2, T_1 \leq t \leq T_2.$$

Secondly, since $\alpha(s)$ is an asymptotic curve on the surface, by Eqn. 1 there exists a parameter $t_0 \in [T_1, T_2]$ such that $\left\langle \frac{\partial n}{\partial s}(s, t_0), T(s) \right\rangle = 0$.

Theorem: A spacelike curve $\alpha(s)$ is isoasymptotic on a spacelike surface $P(s, t)$ if

$$\begin{cases} u(s, t_0) = v(s, t_0) = w(s, t_0) \equiv 0, \\ \frac{\partial w}{\partial t}(s, t_0) \equiv 0, \end{cases} \quad (3)$$

is satisfied.

Proof: Let $\alpha(s)$ be a spacelike curve on a spacelike surface $P(s, t)$. If $\alpha(s)$ is a parameter curve on this surface, then there exists a parameter $t = t_0$ such that $\alpha(s) = P(s, t_0)$, that is,

$$u(s, t_0) = v(s, t_0) = w(s, t_0) \equiv 0. \quad (4)$$

The normal vector of $P(s, t)$ can be written as

$$n(s, t) = \frac{\partial P(s, t)}{\partial s} \times \frac{\partial P(s, t)}{\partial t}$$

Since

$$\begin{aligned} \frac{\partial P(s, t)}{\partial s} &= \left(1 + \frac{\partial u(s, t)}{\partial s} + \varepsilon \kappa(s) v(s, t) \right) T(s) \\ &\quad + \left(\frac{\partial v(s, t)}{\partial s} + \kappa(s) u(s, t) + \tau(s) w(s, t) \right) N(s) \\ &\quad + \left(\frac{\partial w(s, t)}{\partial s} + \tau(s) v(s, t) \right) B(s), \end{aligned}$$

$$\frac{\partial P(s, t)}{\partial t} = \frac{\partial u(s, t)}{\partial t} T(s) + \frac{\partial v(s, t)}{\partial t} N(s) + \frac{\partial w(s, t)}{\partial t} B(s),$$

the normal vector can be expressed as

$$\begin{aligned} n(s, t) &= \left[\left(\frac{\partial w(s, t)}{\partial s} + \tau(s) v(s, t) \right) \frac{\partial v(s, t)}{\partial t} \right. \\ &\quad \left. - \left(\frac{\partial v(s, t)}{\partial s} + \kappa(s) u(s, t) + \tau(s) w(s, t) \right) \frac{\partial w(s, t)}{\partial t} \right] T(s) \\ &\quad + \left[\left(\frac{\partial w(s, t)}{\partial s} + \tau(s) v(s, t) \right) \frac{\partial u(s, t)}{\partial t} \right. \\ &\quad \left. - \left(1 + \frac{\partial u(s, t)}{\partial s} + \varepsilon \kappa(s) v(s, t) \right) \frac{\partial w(s, t)}{\partial t} \right] \varepsilon N(s) \\ &\quad + \left[\left(\frac{\partial v(s, t)}{\partial s} + \kappa(s) u(s, t) + \tau(s) w(s, t) \right) \frac{\partial u(s, t)}{\partial t} \right. \\ &\quad \left. - \left(1 + \frac{\partial u(s, t)}{\partial s} + \varepsilon \kappa(s) v(s, t) \right) \frac{\partial v(s, t)}{\partial t} \right] \varepsilon B(s) \end{aligned}$$

Thus, if we let

$$\begin{aligned}\varphi_1(s, t_0) &= \left(\frac{\partial w}{\partial s} \frac{\partial v}{\partial t} - \frac{\partial v}{\partial s} \frac{\partial w}{\partial t} \right) (s, t_0), \\ \varphi_2(s, t_0) &= \left(\frac{\partial w}{\partial s} \frac{\partial u}{\partial t} - \left(1 + \frac{\partial u}{\partial s} \right) \frac{\partial w}{\partial t} \right) (s, t_0) \varepsilon, \\ \varphi_3(s, t_0) &= \left(\frac{\partial v}{\partial s} \frac{\partial u}{\partial t} - \left(1 + \frac{\partial u}{\partial s} \right) \frac{\partial v}{\partial t} \right) (s, t_0) \varepsilon,\end{aligned}$$

we obtain

$$n(s, t_0) = \varphi_1(s, t_0)T(s) + \varphi_2(s, t_0)N(s) + \varphi_3(s, t_0)B(s).$$

From Eqn. 1, we know that $\alpha(s)$ is an asymptotic curve if

$$\frac{\partial \varphi_1}{\partial s}(s, t_0) + \varepsilon \kappa(s) \varphi_2(s, t_0) = 0.$$

Since $\kappa(s) = \|\alpha''(s)\| \neq 0$, $\varphi_2(s, t_0) = -\frac{\partial v}{\partial t}(s, t_0)$ and by Eqn. 4 we have $\frac{\partial \varphi_1}{\partial s}(s, t_0) = 0$. Therefore, Eqn. 1 is simplified to

$$\frac{\partial w}{\partial t}(s, t_0) \equiv 0,$$

which completes the proof.

We call the set of surfaces defined by Eqn. 2 and satisfying Eqn. 3 *the family of spacelike surfaces with a common spacelike asymptotic*. Any surface $P(s, t)$ defined by Eqn. 2 and satisfying Eqn. 3 is a member of this family.

In Eqn. 2, marching-scale functions $u(s, t)$, $v(s, t)$ and $w(s, t)$ can be chosen in two different forms:

1) If we choose

$$\begin{cases} u(s, t) = \sum_{j=1}^p a_{1j} l(s)^j U(t)^j, \\ v(s, t) = \sum_{j=1}^p a_{2j} m(s)^j V(t)^j, \\ w(s, t) = \sum_{j=1}^p a_{3j} n(s)^j W(t)^j, \end{cases} \quad (5)$$

then we can simply express the sufficient condition for which the spacelike curve $\alpha(s)$ is an

isoasymptotic on the spacelike surface $P(s, t)$ as

$$\begin{cases} U(t_0) = V(t_0) = W(t_0) = 0, \\ a_{31} = 0 \text{ or } n(s) \equiv 0 \text{ or } \frac{dW}{dt}(t_0) = 0, \end{cases} \quad (6)$$

where $l(s), m(s), n(s), U(t), V(t)$ and $W(t)$ are C^1 functions and $a_{ij} \in \mathbb{R}, i=1, 2, 3, j=1, 2, \dots, p$.

2) If we choose

$$\begin{cases} u(s, t) = f\left(\sum_{j=1}^p a_{1j} l(s)^j U(t)^j\right), \\ v(s, t) = g\left(\sum_{j=1}^p a_{2j} m(s)^j V(t)^j\right), \\ w(s, t) = h\left(\sum_{j=1}^p a_{3j} n(s)^j W(t)^j\right), \end{cases} \quad (7)$$

then we can simply express the sufficient condition for which the spacelike curve is an isoasymptotic on the spacelike surface $P(s, t)$ as

$$\begin{cases} U(t_0) = V(t_0) = W(t_0) = 0 \text{ and } f(0) = g(0) = h(0) = 0, \\ a_{31} = 0 \text{ or } n(s) \equiv 0 \text{ or } h'(0) = 0 \text{ or } \frac{dW}{dt}(t_0) = 0, \end{cases} \quad (8)$$

where $l(s), m(s), n(s), U(t), V(t)$ and $W(t), f, g$ and h are C^1 functions and $a_{ij} \in \mathbb{R}, i=1, 2, 3, j=1, 2, \dots, p$.

Because the parameters $a_{ij}, i=1, 2, 3, j=1, 2, \dots, p$ in Eqns. 5 and 7 control the shape of the surface, these parameters can be adjusted to produce spacelike surfaces with given constraints. The marching-scale functions in Eqns. 5 and 7 are general enough for expressing surfaces with a given curve as an isoasymptotic curve. Furthermore, conditions for different types of marching-scale functions can be obtained from Eqn. 3.

Because there are no constraints related to the given curve in Eqns 6 or 8, a spacelike surface family passing through a given regular arc length curve, acting as both a parametric curve and an asymptotic curve, can always be found by choosing suitable marching-scale functions.

Example 1: Consider the parametric spacelike curve $\alpha(s) = (\cos s, \sin s, 0), 0 \leq s \leq 2\pi$.

We will construct a spacelike surface family possessing the curve $\alpha(s)$ as a spacelike isoasymptotic. It is easy to show that

$$\begin{cases} T(s) = (-\sin s, \cos s, 0), \\ N(s) = (-\cos s, -\sin s, 0), \\ B(s) = (0, 0, 1). \end{cases}$$

If we choose $u(s, t) \equiv 0$, $v(s, t) = \cos t + \sum_{j=2}^p a_{2j} \cos^j t$, $w(s, t) = \sum_{j=1}^p a_{3j} (1 + \sin t)^j$ and $t_0 = \frac{3\pi}{2}$ then Eqn. 6 is satisfied. Thus, we obtain a member of this family as

$$P(s, t) = \left(\left(1 - \cos t - \sum_{j=2}^4 \frac{1}{2} (\cos t)^j \right) \cos s, \right. \\ \left. \left(1 - \cos t - \sum_{j=2}^4 \frac{1}{2} (\cos t)^j \right) \sin s, \right. \\ \left. \sum_{j=1}^4 \frac{1}{2} (1 + \sin t)^j \right),$$

where $0 \leq s \leq 2\pi$, $4 \leq t \leq 5$ (Fig. 1).

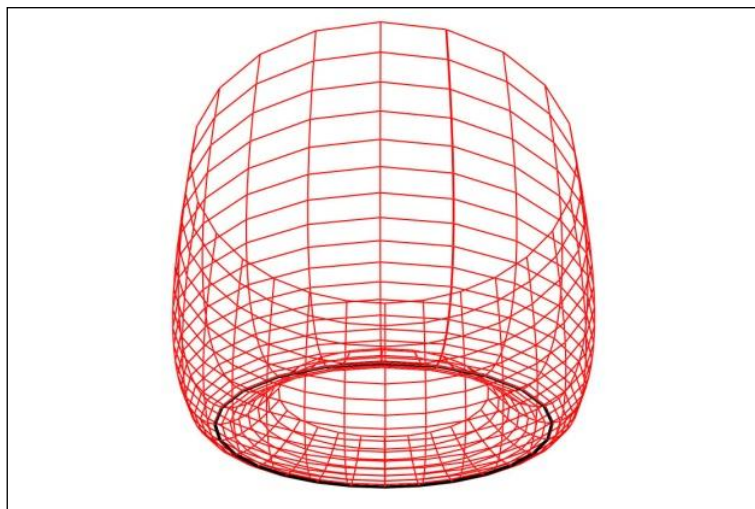


Figure 1. A member of spacelike surface family and its common spacelike asymptotic.

4. TIMELIKE SURFACES WITH A COMMON SPACELIKE OR TIMELIKE ASYMPTOTIC

Let $P(s, t)$ be a parametric timelike surface. The surface is defined by a given curve $\alpha = \alpha(s)$ as follows:

$$\begin{cases} P(s,t) = \alpha(s) + u(s,t)T(s) + v(s,t)N(s) + w(s,t)B(s), \\ L_1 \leq s \leq L_2, T_1 \leq t \leq T_2, \end{cases} \quad (9)$$

where $u(s,t)$, $v(s,t)$ and $w(s,t)$ are C^1 functions and $\{T(s), N(s), B(s)\}$ is the Frenet frame associated with the curve $\alpha(s)$.

Similar computation shows that the conditions 3, 6 and 8 are valid for a curve to be both isoparametric and asymptotic on timelike surfaces. We call the set of surfaces defined by Eqn. 9 and satisfying Eqn. 3 *the family of timelike surfaces with a common timelike asymptotic*. Any surface defined by Eqn. 9 and satisfying Eqn. 3 is a member of this family.

Now let us give some examples for timelike surfaces with a common (spacelike or timelike) asymptotic curve:

Example 2: Given the parametric timelike curve $\alpha(s) = (\cosh s, 0, \sinh s)$, where $-2 \leq s \leq 2$ it is easy to show that

$$\begin{cases} T(s) = (\sinh s, 0, \cosh s), \\ N(s) = (\cosh s, 0, \sinh s), \\ B(s) = (0, 1, 0). \end{cases}$$

If we choose $u(s,t) \equiv 0$, $v(s,t) = \sin t$, $w(s,t) = st^2$ and $t_0 = 0$ then Eqn. 6 is satisfied. By putting these functions into Eqn. 9, we obtain the following timelike surface passing through the common asymptotic $\alpha(s)$ as

$$P(s,t) = (\cosh s + (\sin t)\cosh s, st^2, \sinh s + (\sin t)\sinh s),$$

where $-2 \leq s \leq 2$, $-1 \leq t \leq 1$ (Fig. 2).

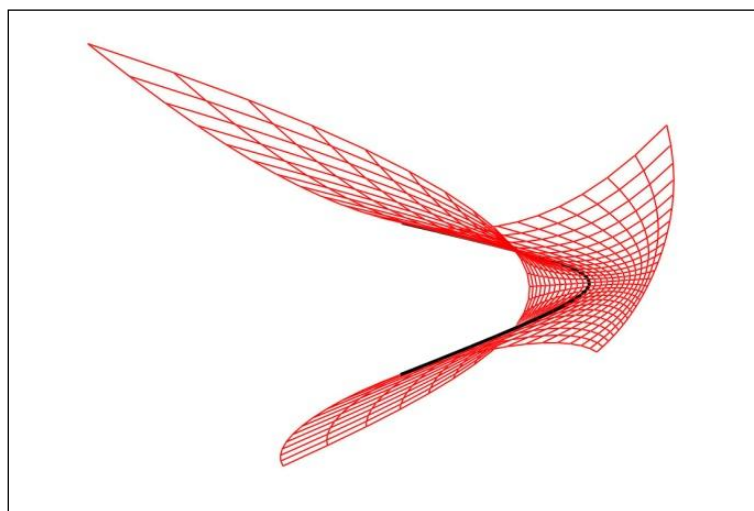


Figure 2. A member of timelike surface family and its common timelike asymptotic.

Example3: Suppose we are given a parametric spacelike curve $\alpha(s) = (\cos s, \sin s, 0)$, where $0 \leq s \leq 2\pi$. We will construct a family of spacelike surfaces sharing the curve $\alpha(s)$ as a common spacelike isoasymptotic. It is obvious that

$$\begin{cases} T(s) = (-\sin s, \cos s, 0), \\ N(s) = (-\cos s, -\sin s, 0), \\ B(s) = (0, 0, 1). \end{cases}$$

By choosing marching-scale functions as

$$u(s, t) \equiv 0, \quad v(s, t) = \sin \left(\sum_{j=1}^p a_{1j} (\sinh t)^j \right),$$

$$w(s, t) = \sin \left(\sum_{j=1}^p a_{2j} (1 - \cosh t)^j \right),$$

where $a_{1j}, a_{2j} \in \mathbb{R}$ and letting $t_0 = 0$, Eqn. 8 is satisfied. Thus, we obtain the surface family with a common spacelike asymptotic curve $\alpha(s)$ as

$$P(s, t) = \begin{pmatrix} \left(1 - \sin \left(\sum_{j=1}^p a_{1j} (\sinh t)^j \right) \right) \cos s, \\ \left(1 - \sin \left(\sum_{j=1}^p a_{1j} (\sinh t)^j \right) \right) \sin s, \\ \sin \left(\sum_{j=1}^p a_{2j} (1 - \cosh t)^j \right) \end{pmatrix},$$

where $0 \leq s \leq 2\pi$, $0 \leq t \leq \frac{1}{2}$. Letting $p = 4$, $a_{1j}, a_{2j} = 1, \forall j$, we obtain the following member of this family (Fig. 3).

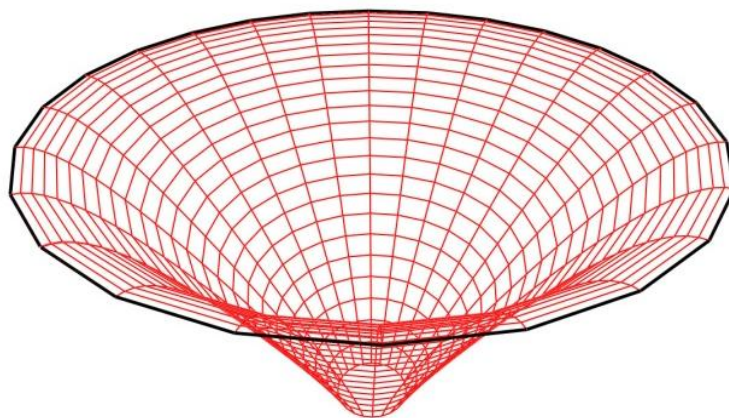


Figure 3. A member of spacelike surface family and its common spacelike asymptotic.

Now, we give some special examples. We construct all minimal timelike surfaces (i.e. helicoid of the 1st, 2nd and 3rd kind and the conjugate surface of Enneper of the 2nd kind, [23]) as members of timelike surface family with a common asymptotic curve.

Example4 (The helicoid of the 1st kind): Let $\alpha(s) = \left(\frac{4}{9} \cos 3s, \frac{4}{9} \sin 3s, \frac{5}{3}s\right)$ be a timelike curve, where $0 \leq s \leq 2\pi$. It is easy to show that

$$\begin{cases} T(s) = \left(-\frac{4}{3} \sin 3s, \frac{4}{3} \cos 3s, \frac{5}{3}\right), \\ N(s) = (-\cos 3s, -\sin 3s, 0), \\ B(s) = \left(\frac{5}{3} \sin 3s, -\frac{5}{3} \cos 3s, -\frac{4}{3}\right). \end{cases}$$

If we choose $u(s,t) = w(s,t) \equiv 0$ and $v(s,t) = t$ and $t_0 = 0$ then Eqn. 3 is satisfied. Thus, we get the helicoid of the 1st kind as a member of timelike minimal surface family with common timelike asymptotic as

$$P(s,t) = \left(\left(\frac{4}{9} - t\right) \cos 3s, \left(\frac{4}{9} - t\right) \sin 3s, \frac{5}{3}s \right),$$

where $0 \leq s \leq 2\pi$, $-1 \leq t \leq 1$ (Fig. 4).

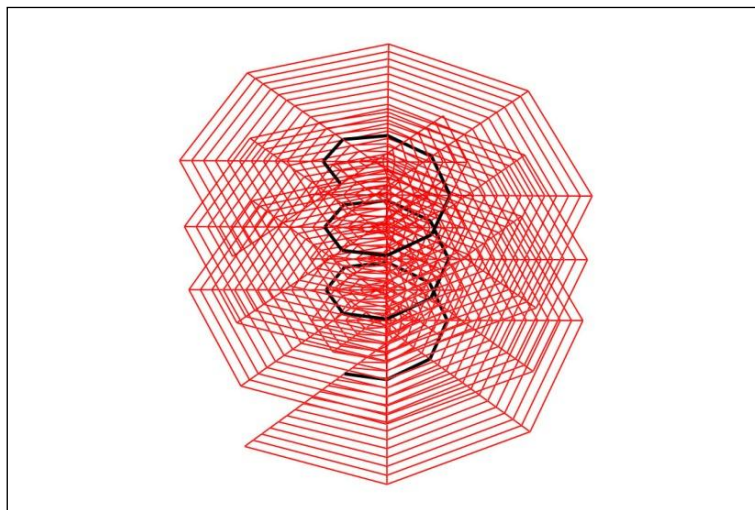


Figure 4. A member of timelike minimal surface family and its common timelike asymptotic (The helicoid of the 1st kind).

Example 5 (The helicoid of the 2nd kind): Let $\alpha(s) = \left(-\frac{5}{9} \cosh 3s, \frac{4}{3}s, -\frac{5}{9} \sinh 3s\right)$ be a timelike curve, where $-1 \leq s \leq 1$. It is easy to show that

$$\begin{cases} T(s) = \left(-\frac{5}{3} \sinh 3s, \frac{4}{3}, -\frac{5}{3} \cosh 3s\right), \\ N(s) = (-\cosh 3s, 0, -\sinh 3s), \\ B(s) = \left(-\frac{4}{3} \sinh 3s, \frac{5}{3}, -\frac{4}{3} \cosh 3s\right). \end{cases}$$

If we choose $u(s,t) = w(s,t) \equiv 0$ and $v(s,t) = t$ and $t_0 = 0$, then Eqn. 3 is satisfied. Thus, we obtain the helicoid of the 2nd kind as a member of timelike minimal surface family with common timelike asymptotic as

$$P(s,t) = \left(\left(-\frac{5}{9} - t \right) \cosh 3s, \frac{4}{3} s, \left(-\frac{5}{9} - t \right) \sinh 3s \right),$$

where $-1 \leq s \leq 1$, $-1 \leq t \leq 1$ (Fig. 5).

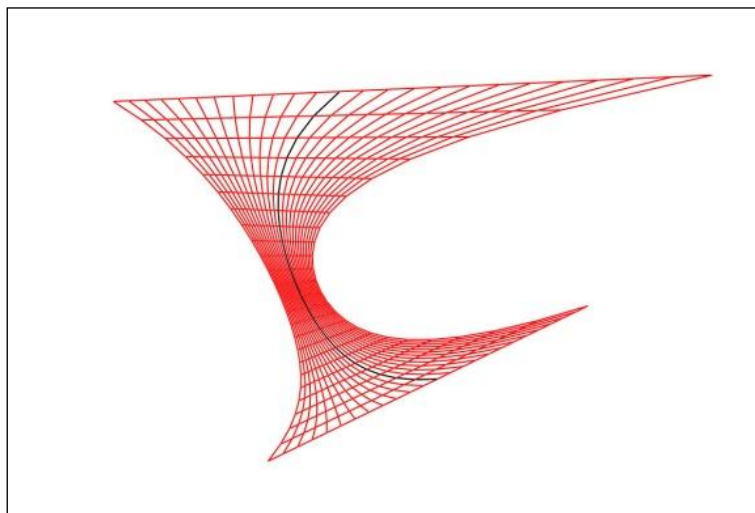


Figure 5. A member of timelike minimal surface family and its common timelike asymptotic (The helicoid of the 2nd kind).

Example 6 (The helicoid of the 3rd kind): Let $\alpha(s) = \left(-\frac{3}{25} \sinh 5s, \frac{4}{5} s, -\frac{3}{25} \cosh 5s\right)$ be a spacelike curve, where $-1 \leq s \leq 1$. It is easy to show that

$$\begin{cases} T(s) = \left(-\frac{3}{5} \cosh 5s, \frac{4}{5}, -\frac{3}{5} \sinh 5s\right), \\ N(s) = (-\sinh 5s, 0, -\cosh 5s), \\ B(s) = \left(-\frac{4}{5} \cosh 5s, -\frac{3}{5}, -\frac{4}{5} \sinh 5s\right). \end{cases}$$

If we choose $u(s,t) = w(s,t) \equiv 0$ and $v(s,t) = t$ and $t_0 = 0$, then Eqn. 3 is satisfied. Thus, we obtain the helicoid of the 3rd kind as a member of timelike minimal surface family with common timelike asymptotic as

$$P(s,t) = \left(\left(-\frac{3}{25} - t \right) \sinh 5s, \frac{4}{5}s, \left(-\frac{3}{25} - t \right) \cosh 5s \right),$$

where $-1 \leq s \leq 1, -1 \leq t \leq 1$ (Fig. 6).

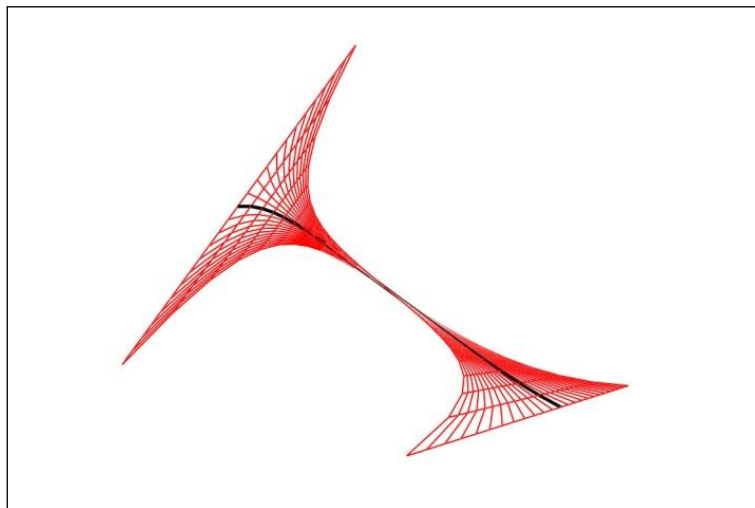


Figure 6. A member of timelike minimal surface family and its common spacelike asymptotic (The helicoid of the 3rd kind).

Example 7 (The conjugate surface of Enneper of the 2nd kind):

Let $\alpha(s) = \left(\frac{s^2}{2}, -\frac{s^3}{6}, s - \frac{s^3}{6} \right)$ be a timelike curve, where $-2 \leq s \leq 2$. It is easy to show that

$$\begin{cases} T(s) = \left(s, -\frac{s^2}{2}, 1 - \frac{s^2}{2} \right), \\ N(s) = (1, -s, -s), \\ B(s) = \left(-s, -\frac{s^2}{2} - 1, -\frac{s^2}{2} \right). \end{cases}$$

If we choose $u(s,t) = w(s,t) \equiv 0$ and $v(s,t) = t$ and $t_0 = 0$, then Eqn. 3 is satisfied. Thus, we obtain the conjugate surface of Enneper of the 2nd kind as a member of timelike minimal surface family with common timelike asymptotic as

$$P(s, t) = \left(\frac{s^2}{2} + t, -\frac{s^3}{6} - st, s - \frac{s^3}{6} - st \right),$$

where $-2 \leq s \leq 2$, $-1 \leq t \leq 1$ (Fig. 7).

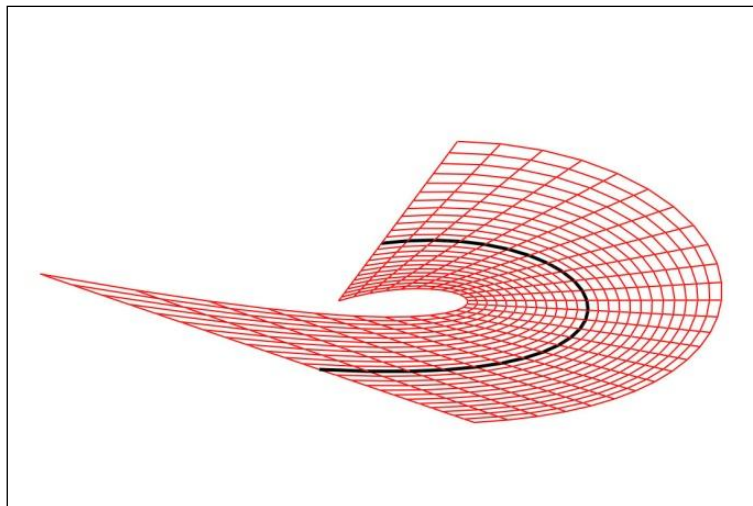


Figure 7. A member of timelike minimal surface family and its common timelike asymptotic (The conjugate surface of Enneper of the 2nd kind).

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