*ORIGINAL PAPER*

# THE TIMELIKE RULED SURFACES ACCORDING TO TYPE -2 BISHOP FRAME IN MINKOWSKI 3-SPACE

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***Abstract.*** *In this paper, the timelike ruled surfaces generated by vectors of type-2 bishop frame were investigated. Using this frame, the necessary and sufficient conditions when the ruled surfaces are developable were obtained and some new results and theorems related to be the asymptotic curve, the geodesic curve of the base curve on the ruled surfaces were gived.Also, the gaussian and the mean curvatures of timelike ruled surfaces were calculated.*

***Keywords:*** *Timelike ruled surfaces, Curves, Bishop Frame.*

## 1. INTRODUCTION

A surface is said to be “ruled” if it is generated by moving a straight line constinuously in Minkowski space. Different elements on a rigid body form different orbits. In general, a directed solid line on a moving solid body forms a ruled surface. Ruled surfaces are one of the simplest objects in geometric modeling.

The Frenet frame on the curve is a coordinate frame attached to the curve that helps describe the geometry of the curve. The useful property for curves parameterized by arc length is the Serret Frenet formulas. These formulas show the derivatives with respect to arc length of the Frenet frame as a function of the current Frenet frame, curvature and torsion. Another usable the frame of curve is Bishop Frame. This frame, which is also called alternative or parallel frame of the curves, was introduced by L. R. Bishop in 1975 by means of parallel vector fields [1]. A practical application of Bishop frames is that they are used in the area of Biology and Computer Graphics. For example, it may be possible to compute information about the shope of sequences of DNA using a curve defined by the Bishop frame. The Bishop frame may also provide a new way to control virtual cameras in computer animations [6]. In recent years several authors have used this frame in Euclidean and Minkowski space, see [2, 3, 8]. Yılmaz and Turgut have introduced a new version of Bishop frame using a common vector field as binormal vector field of a regular frame and called this frame type-2 Bishop frame in Euclidean space. Then, the type-2 Bishop frame has been research topic for authors [2-4, 9]. Later, Savcı has introduced type-2 Bishop frame of a regular timelike curve in Minkowski space [7]. Furthermore, he has given the type-2 Bishop spherical images in Minkowski space.

The objective of the study in this paper is to investigate the timelike ruled surfaces which are generated from the type-2 Bishop vectors. We obtained the necessary and sufficient conditions when the timelike ruled surfaces are developable and we give some new results and theorems related to be the asymptotic curve, the geodesic curve of the base curve on the

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ruled surfaces. Also, we have calculated the gaussian and the mean curvatures of timelike ruled surfaces.

## 2. PRELIMINARIES

The Minkowski three dimensional space $E_1^3$ is a real vector space $IR^3$ endowed with the Standard flat Lorentzian metric given by

$$g = -dx_1 + dx_2 + dx_3 \tag{1}$$

where $(x_1, x_2, x_3)$ is rectangular coordinate system of $E_1^3$. Since g is an indefinite metric. Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be arbitrary an vectors in $E_1^3$, the Lorentzian cross product of u and v defined by

$$u \times v = -\det \begin{bmatrix} -i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}. \tag{2}$$

Recall that a vector $u \in E_1^3$ can have one of three Lorentzian characters: it can be spacelike if $g(u,u) > 0$ or $u = 0$; timelike if $g(u,u) < 0$ and null (lightlike) if $g(u,u) = 0$ for $u \neq 0$. Similarly, an arbitrary curve $r = r(s)$ in $E_1^3$ can locally be spacelike, timelike or null (lightlike) if all of its velocity vector $r' = dr/ds$ are respectively spacelike, timelike or null (lightlike), for every $s \in I \subset IR$. The pseudo-norm of an arbitrary vector $\alpha \in E_1^3$ is given by

$$\|\alpha\| = \sqrt{|g(\alpha, \alpha)|}. \tag{3}$$

The curve $\alpha = \alpha(s)$ is called a unit speed curve if velocity vector $\alpha'$ is unit i.e, $\|\alpha'\| = 1$. For vectors $u, v \in E_1^3$ it is said to be orthogonal if and only if $g(u,v) = 0$. Denote by $\{T, N, B\}$ the moving Serret –Frenet frame along the curve $\alpha = \alpha(s)$ in the space $E_1^3$. For an arbitrary timelike curve $\alpha = \alpha(s)$ in $E_1^3$, the following Serret-Frenet formulea are given in as follows

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{4}$$

where

$$g(T,T) = -1, g(B,B) = g(N,N) = 1, T(s) = \alpha'(s), N(s) = T'(s)/\kappa(s), B(s) = T(s) \times N(s)$$

and first curvature and second curvature $\kappa(s), \tau(s)$ respectively.

$$\kappa(s)=\|\alpha''\|,\tau(s)=\det(\alpha',\alpha'',\alpha''')/\kappa^2, \text{ [5].}$$

Denoted by $\{\Omega_1,\Omega_2,B\}$ moving Type-2 Bishop frame along to timelike curve $\alpha: I\subset IR\rightarrow E_1^3$ in the Minkowski 3- space $E_1^3$. For an arbitrary timelike curve $\alpha=\alpha(s)$ in 3-space $E_1^3$,the following Type-2 Bishop formulea are given by

$$\begin{bmatrix}\Omega_1'\\ \Omega_2'\\ B'\end{bmatrix}=\begin{bmatrix}0 & 0 & \xi_1\\ 0 & 0 & \xi_2\\ \xi_1 & \xi_2 & 0\end{bmatrix}\begin{bmatrix}\Omega_1\\ \Omega_2\\ B\end{bmatrix} \tag{5}$$

where $g(\Omega_1,\Omega_1)=-1, g(\Omega_2,\Omega_2)=g(B,B)=1$ and $g(\Omega_1,\Omega_2)=g(\Omega_1,B)=g(\Omega_2,B)=0$ and $\xi_1=\tau(s)\cosh\theta(s),\xi_2=\tau(s)\sinh\theta(s)$ as type-2 Bishop curvature of the curve $\alpha=\alpha(s)$ in $E_1^3$ . If $\Omega_1$ timelike, $\Omega_2$ and $B$ spacelike vectors, [7].

The relations between Frenet and type-2 Bishop frames are

$$\begin{bmatrix}T\\ N\\ B\end{bmatrix}=\begin{bmatrix}\sinh\theta & -\cosh\theta & 0\\ \cosh\theta & \sinh\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\Omega_1\\ \Omega_2\\ B\end{bmatrix} \tag{6}$$

where $\theta$ is the angle between the vectors $N$ and $\Omega_1$, [7].

The trace of an $\vec{X}$ oriented line along a space curve $\alpha(s)$ is generally a ruled surface. A parametric equation of this ruled surface generated by

$$\phi(s,v)=\alpha(s)+v\vec{d}(s),\quad s,v\in I\subset IR \tag{7}$$

where $\vec{d}$ is the unit direction vector of $\vec{X}$ oriented line.

The standart unit normal vector field of the ruled surface $\phi$ can be given by

$$n=\frac{\phi_s\times\phi_v}{|\phi_s\times\phi_v|} \tag{8}$$

where $\phi_s=\dfrac{\partial\phi(s,v)}{\partial s}$ and $\phi_v=\dfrac{\partial\phi(s,v)}{\partial v}$.

The gaussian curvature $K$ and the mean curvature $H$ of the ruled surface $\phi$ are given by

$$K(s,v)=-\varepsilon\frac{LN-M^2}{EG-F^2},\quad H(s,v)=-\varepsilon\frac{GL+EN-2FM}{2(EG-F^2)} \tag{9}$$

where $\varepsilon=\begin{cases}-1, & n \text{ timelike vector}\\ 1, & n \text{ spacelike vector}\end{cases}$.

A developable surface is a surface with zero Gaussian curvature.

**Definition 1.** The distribution parameter(or drall) of the ruled surface is given as [5]:

$$P_{\vec{d}} = \frac{\det\left(\alpha'(s), \vec{d}(s), \vec{d}'(s)\right)}{\left|\vec{d}'(s)\right|^2}.$$

**Lemma 2.** The ruled surface Eqn. (7) is developable if and only if, by [5]
$\det\left(\alpha'(s), \vec{d}(s), \vec{d}'(s)\right) = 0$.

**Theorem 3.** A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable surface [10].

**Definition 4.** For a curve lying on a surface, the following are well-known:

1. $(\alpha)$ curve is an asymptotic line of surface if and only if normal curvature $k_n$ vanishes.
2. $(\alpha)$ curve is an geodesic line of surface if and only if geodesic curvature $k_g$ vanishes [5].

## 3. THE TIMELIKE RULED SURFACES ACCORDING TO TYPE-2 BISHOP FRAME

Let $(\alpha)$ be a regular timelike curve and the set $\{\Omega_1, \Omega_2, B\}$ be the type-2 Bishop frame along to timelike curve $(\alpha)$. Then the parametric representations of the timelike ruled surfaces $\phi_{\Omega_2}$, $\phi_B$ are

$$\begin{aligned} \phi_{\Omega_2}(s,v) &= \alpha(s) + v\Omega_2(s) \\ \phi_B(s,v) &= \alpha(s) + vB(s) \end{aligned} \tag{10}$$

and these ruled surfaces are called second rectifying timelike surface and binormal timelike surface according to type-2 Bishop frame, respectively.

### *3.1. THE SECOND RECTİFYİNG TİMELİKE RULED SURFACE*

The distrubition parameter of the timelike ruled surface $\phi_{\Omega_2}(s,v)$ is

$$P_{\Omega_2} = -\frac{\sinh\theta}{\xi_2}. \tag{11}$$

The unit normal vector to timelike ruled surface $\phi_{\Omega_2}(s,v)$ is given by

$$\overline{n}_{\Omega_2} = \frac{\phi_{\Omega_2^s} \times \phi_{\Omega_2^v}}{\left|\phi_{\Omega_2^s} \times \phi_{\Omega_2^v}\right|}. \tag{12}$$

Thus, the unit vector to ruled surface $\phi_{\Omega_2}(s,v)$ is

$$k_{n_{\Omega_2}} = \left\langle \overline{n}'_{\Omega_2}, \Omega_2 \right\rangle = \frac{\xi_2 \sinh\theta}{\sqrt{\sinh^2\theta + v^2\xi_2^2}}. \tag{13}$$

The normal curvature of the base curve is

$$k_{n_{\Omega_2}} = \left\langle \overline{n}'_{\Omega_2}, \Omega_2 \right\rangle = \frac{\xi_2 \sinh\theta}{\sqrt{\sinh^2\theta + v^2\xi_2^2}}. \tag{14}$$

**Corollary 5.** Timelike ruled surface $\phi_{\Omega_2}(s,v)$ is developable if and only if the base curve $(\alpha)$ is asymptotic curve on the timelike ruled surface $\phi_{\Omega_2}(s,v)$.

**Corollary 6.** The base curve $(\alpha)$ is geodesic curve on the ruled surface $\phi_{\Omega_2}(s,v)$ if and only if $\theta = 0$.

**Theorem 7.** The Gauss curvature and mean curvature of timelike ruled surface $\phi_{\Omega_2}(s,v)$ are

$$K_{\Omega_2}(s,v) = \frac{\xi_2{}^2 \sinh^2\theta}{v^2\xi_2{}^2 - \sinh^2\theta},\ H_{\Omega_2}(s,v) = \frac{\xi_2 \sinh\theta\cosh\theta + \xi_1 \sinh^2\theta + v^2\xi_1\xi_2{}^2 + v(\xi_2 \sinh\theta)'}{2\left(v^2\xi_2{}^2 - \sinh^2\theta\right)}.$$

*Proof.* From Eqn.(9) by a direct calculation we can get the conclusion of this theorem, where $\varepsilon = \begin{cases} -1, & \overline{n}_{\Omega_2} \text{ timelike vector} \\ 1, & \overline{n}_{\Omega_2} \text{ spacelike vector} \end{cases}$ and $\overline{n}_{\Omega_2}$ is the unit normal vector of $\phi_{\Omega_2}(s,v)$. The first fundamental quantities of $\phi_{\Omega_2}(s,v)$ are

$$E_{\Omega_2} = 1 + v^2\xi_2^2,\ \ F_{\Omega_2} = -\cosh\theta,\ G_{\Omega_2} = 1.$$

The second fundamental quantities are

$$L_{\Omega_2} = \xi_1 \sinh^2\theta + v\xi_2\left(\theta'\cosh\theta + v\xi_1\xi_2\right) - \xi_2 \cosh\theta\sinh\theta + \xi_2{}'\sinh\theta,$$
$$M_{\Omega_2} = \xi_2 \sinh\theta,\ N_{\Omega_2} = 0.$$

The Gauss curvature and the mean curvature of $\phi_{\Omega_2}(s,v)$ are

$$K_{\Omega_2}(s,v) = \frac{\xi_2{}^2 \sinh^2\theta}{v^2\xi_2{}^2 - \sinh^2\theta},\ H_{\Omega_2}(s,v) = \frac{\xi_2 \sinh\theta\cosh\theta + \xi_1 \sinh^2\theta + v^2\xi_1\xi_2{}^2 + v(\xi_2 \sinh\theta)'}{2\left(v^2\xi_2{}^2 - \sinh^2\theta\right)}.$$

*3.2. THE BINORMAL TIMELIKE RULED SURFACE*

The distrubition parameter of the binormal timelike ruled surface $\phi_B(s,v)$ is

$$P_B = \frac{\xi_1 \cosh\theta + \xi_2 \sinh\theta}{\xi_1^2 + \xi_2^2}. \tag{15}$$

The unit normal vector to timelike ruled surface $\phi_B(s,v)$ is given by

$$\bar{n}_B = \frac{\phi_{B^s} \times \phi_{B^v}}{\left|\phi_{B^s} \times \phi_{B^v}\right|}. \tag{16}$$

Thus, from Eqn. (16) the unit normal vector to ruled surface timelike $\phi_B(s,v)$ is

$$\bar{n}_B(s,v) = \frac{(v\xi_2 - \cosh\theta)\Omega_1 - (v\xi_1 + \sinh\theta)\Omega_2}{\sqrt{(v\xi_1 + \sinh\theta)^2 - (v\xi_2 - \cosh\theta)^2}}. \tag{17}$$

The normal curvature of the base curve is

$$k_{nB} = \langle \bar{n}_B', B \rangle = -\frac{\xi_1 \cosh\theta + \xi_2 \sinh\theta}{\sqrt{(v\xi_1 + \sinh\theta)^2 - (v\xi_2 - \cosh\theta)^2}}. \tag{18}$$

**Corollary 8.** The base curve $(\alpha)$ is asymptotic curve and geodesic curve on the ruled surface $\phi_B(s,v)$ if and only if $\frac{\xi_1}{\xi_2} = -\tanh\theta$.

**Theorem 9.** The Gauss curvature and the mean curvature of timelike ruled surface $\phi_B(s,v)$ are

$$K_B(s,v) = -\frac{(\xi_1 \cosh\theta - \xi_2 \sinh\theta - 2v\xi_1\xi_2)^2}{(v\xi_2 - \cosh\theta)^2 - (v\xi_1 + \sinh\theta)^2}$$

$$H_B(s,v) = -\frac{(\cosh\theta - v\xi_2)(\theta'\cosh\theta + v\xi_1') - (\sinh\theta + v\xi_1)(-\theta'\sinh\theta + v\xi_2')}{2\left[(\cosh\theta - v\xi_2)^2 - (\sinh\theta + v\xi_1)^2\right]}.$$

*Proof.* From Eqn.(9) by a direct calculation we can get the conclusion of this theorem, where $\varepsilon = \begin{cases} -1, & \bar{n}_B \text{ timelike vector} \\ 1, & \bar{n}_B \text{ spacelike vector} \end{cases}$ and $\bar{n}_B$ is the unit normal vector of $\phi_B(s,v)$. The first fundamental quantities of $\phi_B(s,v)$ are

$$E_B = (\cosh\theta - v\xi_2)^2 - (\sinh\theta + v\xi_1)^2,\ F_B = 0,\ G_B = 1.$$

The second fundamental quantities are

$$L_B = (\cosh\theta - v\xi_2)(\theta'\cosh\theta + v\xi_1') - (\sinh\theta + v\xi_1)(-\theta'\sinh\theta + v\xi_2'),$$

and

$$M_B = \xi_1\cosh\theta - \xi_2\sinh\theta - 2v\xi_1\xi_2, \quad N_B = 0.$$

The Gauss curvature and the mean curvature of $\phi_B(s,v)$ are

$$K_B(s,v) = -\frac{(\xi_1\cosh\theta - \xi_2\sinh\theta - 2v\xi_1\xi_2)^2}{(v\xi_2 - \cosh\theta)^2 - (v\xi_1 + \sinh\theta)^2}$$

and

$$H_B(s,v) = -\frac{(\cosh\theta - v\xi_2)(\theta'\cosh\theta + v\xi_1') - (\sinh\theta + v\xi_1)(-\theta'\sinh\theta + v\xi_2')}{2\left[(\cosh\theta - v\xi_2)^2 - (\sinh\theta + v\xi_1)^2\right]}$$

**Example 10.** Let

$$\alpha(s) = \left(-\sqrt{2}\sinh s, s, \sqrt{2}\cosh s\right)$$

be a unit speed timelike curve. Then type-2 Bishop frame is easy to show that

$$\Omega_1(s) = \left(-\sqrt{2}\sinh(s+\theta), \sinh\theta, \sqrt{2}\cosh(s+\theta)\right)$$

$$\Omega_2(s) = \left(-\sqrt{2}\cosh(s+\theta), -\cosh\theta, -\sqrt{2}\sinh(s+\theta)\right)$$

$$B(s) = \left(\sqrt{2}\cosh s, 2, \sqrt{2}\sinh s\right).$$

i) If we take $\theta = 0$, then the corollary (6) is satisfied. Thus, we obtain geodesic curve $(\alpha)$ on the ruled surface $\phi_{\Omega_2}(s,v)$,where $-2 \le s \le 2, -2 \le v \le 2$ (Fig. 1).

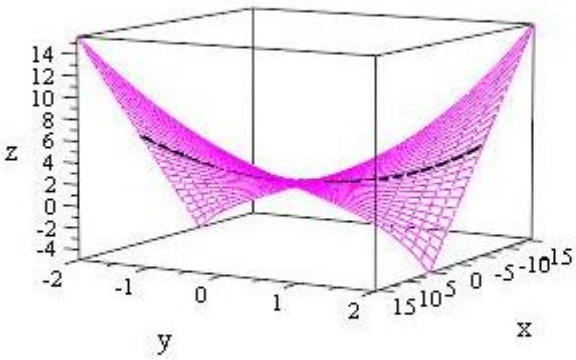


**Figure 1. Geodesic curve (α) on the ruled surface $\phi_{\Omega 2}$ (s, v).**

ii) If we take $\theta = {}^{i\pi}\!/_{4}$, then the corollary (8) is satisfied. Thus, we obtain asymptotic and geodesic curve $(\alpha)$ on the ruled surface $\phi_B(s,v)$ where $-2 \le s \le 2,\ -2 \le v \le 2$ (Fig. 2).

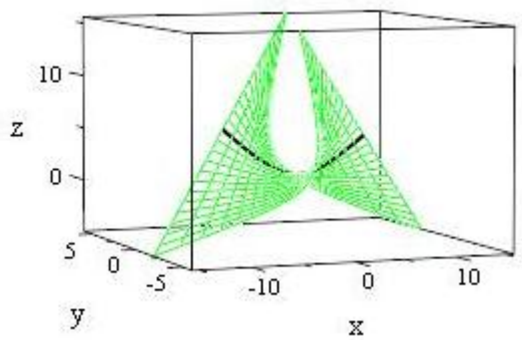


**Figure 2. Asimptotic and geodesic curve (α) on the ruled surface $\phi_B$ (s, v).**

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