ORIGINAL PAPER

SEMI GROUP PROPERTY USING GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. *In this paper the authors shown that the property*

$$E^{\gamma}_{\alpha,\beta}((a(s+t)^{\alpha\beta}) = E^{\gamma}_{\alpha,\beta}(as^{\alpha\beta})\,E^{\gamma}_{\alpha,\beta}(at^{\alpha\beta}); s,t \geq 0; a \in R; \alpha,\beta,\gamma > 0$$

is true only when $\alpha = \beta = \gamma = 1$ *and* $a = 0, \beta = 1$ *or* $\beta = 2$. *Moreover, a new equality on* $E^{\gamma}_{\alpha,\beta}(at^{\alpha\beta})$ *is developed, whose limit state as* $\alpha \to 1$, $\gamma \to 1$ *and* $\beta > \alpha$ *is just the above proper and if* $\beta = 1$, *then the result is the same as in [15]. Also, it is proved that this equality is the characteristic of the function* $t^{\beta-1} E^{\gamma}_{\alpha,\beta}(a\alpha t)$. *Finally, we showed that all results in [15] are special cases of our results when* $\beta = \gamma = 1$.

Keywords: *Generalized Mittag–Leffler function, Caputo fractional derivative, Semi group property.*

Mathematics Subject Classification*: 34A12.*

1. INTRODUCTION

As a result of researchers and scientists increasing interest in pure as well as applied mathematics in non-conventional models, particularly those using fractional calculus, Mittag–Leffler functions have recently caught the interest of the scientific community. Focusing on the theory of the generalized Mittag–Leffler functions, the present volume offers a self-contained, comprehensive treatment, ranging from rather elementary matters to the latest research results. In addition to the theory the authors devote some sections of the work to the applications, treating various situations and processes in viscoelasticity, physics, hydrodynamics, diffusion and wave phenomena, as well as stochastics. In particular the generalized Mittag-Leffler functions allow us to describe phenomena in processes that progress or decay too slowly to be represented by classical functions like the exponential function and its successors. The three parameter generalized Mittag–Leffler function is such a three parameter function defined in the complex plane C by

$$E^{\gamma}_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k x^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha > 0$ is the parameter and Γ the Gamma function [16]. It was originally introduced by

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Prabhakar [17] in 1971. Obviously, the exponential function e^z is a particular generalized Mittag–Leffler function with the specified parameter $\alpha=\beta=\gamma=1$, or in other words, the generalized Mittag–Leffler function is the parameterized exponential function. In recent years the generalized Mittag–Leffler function has caused extensive interest among scientists, engineers and applied mathematicians, due to its role played in investigations of fractional differential equations [1, 3, 8, 11, 13, 14]. A large of its properties have been proved (e.g. [4-7, 9, 18, 19]). In this paper we show that the following property

$$E^{\gamma}_{\alpha,\beta}((\mathrm{a}(\mathrm{s}+\mathrm{t})^{\alpha\beta}) = E^{\gamma}_{\alpha,\beta}(\mathrm{as}^{\alpha\beta})\,E^{\gamma}_{\alpha,\beta}(\mathrm{at}^{\alpha\beta}); \mathrm{s},\mathrm{t}\ge 0; \mathrm{a}\in \mathrm{R}; \alpha,\beta,\gamma>0$$

is true only when $\alpha=\beta=\gamma=1$ and $a=0, \beta=1$ or $\beta=2$. Moreover, a new equality on $E^{\gamma}_{\alpha,\beta}(\mathrm{at}^{\alpha\beta})$ is developed, whose limit state as $\alpha\to 1$, $\gamma\to 1$ and $\beta>\alpha$ is just the above proper and if $\beta=1$, then the result is the same as in [15]. Also, it is proved that this equality is the characteristic of the function $\mathrm{t}^{\beta-1}\,E^{\gamma}_{\alpha,\beta}(\mathrm{a}\alpha\mathrm{t})$. To this purpose, the following properties of Mittag–Leffler function and Caputo's fractional derivative are needed:

The Laplace transform of Caputo's derivative is given by[10]

$$\overline{D_t^{\alpha} f(t)}(p) = p^{\alpha}\,\overline{f(t)} - \sum_{k=0}^{n-1} p^{\alpha-k-1} f^{(k)}(0)$$

where $n-1<\alpha\le n$, $\overline{D_t^{\alpha} f(t)}(p)$ and $p^{\alpha}\,\overline{f(t)}$ denote the Laplace transform of $D_t^{\alpha} f(t)$ and $f(t)$ respectively.

The Laplace transform of the Mittag–Leffler functions can be derived from the formula[4]

$$\int_0^{\infty} e^{-pt} t^{\beta-1} E^{\gamma}_{\alpha,\beta}(\mathrm{at}^{\alpha})dt = \frac{p^{\alpha-\beta}}{p^{\alpha}-a}\ , \mathrm{Re}(p) > a^{1/\alpha}, a>0,$$

where $\mathrm{Re}(p)$ represents the real part of the complex number p.

2. A NEW EQUALITY CHARACTERISTIC OF SOLUTION FUNCTION

In this section, firstly we show the following general property

$$E^{\gamma}_{\alpha,\beta}((\mathrm{a}(\mathrm{s}+\mathrm{t})^{\alpha\beta}) = E^{\gamma}_{\alpha,\beta}(\mathrm{as}^{\alpha\beta})\,E^{\gamma}_{\alpha,\beta}(\mathrm{at}^{\alpha\beta}); \mathrm{s},\mathrm{t}\ge 0; \mathrm{a}\in \mathrm{R}; \alpha,\beta,\gamma>0$$

is false, to show that, it is sufficient to provide one counter-example. By using the fact $E^{1}_{2,1}(\mathrm{x}^2) = E_{2,1}(\mathrm{x}^2) = E_2(\mathrm{x}^2) = \frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2} = Coshx$, valid for all z, then elementary calculation shows that (1) does not hold for the choice $\alpha=2$, $\beta=\gamma=1, \mathrm{a}=\mathrm{s}=\mathrm{t}=1$ of parameters.

Hence (1) as a general property is false. Secondly we prove (1) is true only when $\alpha=\beta=\gamma=1$ and $a=0$, $\beta=1$ or $\beta=2$. We want to find all $\alpha,\beta,\gamma>0$, $a\in R$ for which

$$E^{\gamma}_{\alpha,\beta}((a(s+t)^{\alpha\beta}) = E^{\gamma}_{\alpha,\beta}(as^{\alpha\beta})\,E^{\gamma}_{\alpha,\beta}(at^{\alpha\beta}); s,t\geq 0$$

If we set $s=t=0$ we can write

$$E^{\gamma}_{\alpha,\beta}(0) = E^{\gamma}_{\alpha,\beta}(0)\,E^{\gamma}_{\alpha,\beta}(0).$$

Knowing that $E^{\gamma}_{\alpha,\beta}(0)=\dfrac{1}{\Gamma(\beta)}$, yields $\Gamma(\beta)=1$ which implies $\beta=1$ or $\beta=2$. So if $a=0$ then (1) is true with $\beta=1$ or $\beta=2$ and this is the only condition, consequently we can say that for any $a\in$ R and $\alpha,\beta,\gamma>0$ the equality (1) may be true only when $\Gamma(\beta)=1$. Therefore, let us assume that $a\neq 0$ and $\Gamma(\beta)=1$. From (1), if we let $s=t$ then (1) becomes
$E^{\gamma}_{\alpha,\beta}((a(2s)^{\alpha\beta}) = E^{\gamma}_{\alpha,\beta}(as^{\alpha\beta})\,E^{\gamma}_{\alpha,\beta}(at^{\alpha\beta}) = (E^{\gamma}_{\alpha,\beta}(as^{\alpha\beta})^2$.

On taking $z=s^{\alpha\beta}$, then the above equation becomes

$$E^{\gamma}_{\alpha,\beta}(az2^{\alpha\beta}) = (E^{\gamma}_{\alpha,\beta}(az\;)^2 \qquad \text{for all } z\geq 0$$

We differentiating w.r.to z and setting $z=0$ gives

$$\frac{a2^{\alpha\beta}}{\Gamma(\alpha+\beta)}=\frac{2a}{\Gamma(\beta)\Gamma(\alpha+\beta)}$$

This implies $2^{\alpha\beta}=2$. Therefore, $\alpha\beta=1$. If $\beta=1$ then $\alpha=1$ and $E_{\alpha,\beta}(z)=ze$, so (1) is true. Now assume that $\beta=2$. Then $\alpha=1/2$.

Again differentiating w.r.to z and putting $z=0$ we obtain

$$\frac{8a^2}{\Gamma(2\alpha+\beta)}=\frac{2a^2}{\Gamma(\alpha+\beta)^2}+\frac{4a^2}{\Gamma(\beta)\Gamma(2\alpha+\beta)}\;.$$

Putting $\alpha=1/2$ and $\beta=2$, we get

$$4=\frac{32}{9\pi}+2\,,$$

which is not true. Therefore, the main result is true only when $\alpha=\beta=1$, and $a=0$, $\beta=1$ or $\beta=2$.

By definition of Caputo derivative it is clear that the Caputo's fractional derivative operator is non-local in the case of non-integer order α. The memory character of Caputo's derivative operator is perhaps the cause leading to the result that $E^{\gamma}_{\alpha,\beta}(at^{\alpha\beta})$, as an eigen function of Caputo's derivative operator does not possess semigroup property that is non-memory. This seems to tell us that any equality relationship involving $E^{\gamma}_{\alpha,\beta}(at^{\alpha\beta})$, $E^{\gamma}_{\alpha,\beta}(as^{\alpha\beta})$ and $E^{\gamma}_{\alpha,\beta}(a(s+t)^{\alpha\beta})$ should be of memory and hence be characterized with integrals. The equality relationship stated in the following theorem which is a generalization of the results given in [15].

Theorem 2.1. For every real a there holds

$$\int_0^t (t-\tau)^{\beta-\alpha-1} f(s+\tau)d\tau = \int_0^t (s+t-\tau)^{\beta-\alpha-1} f(\tau)d\tau$$

$$+\frac{\alpha\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)}\int_0^s\int_0^t f(\tau_1)f(\tau_2)(t+s-\tau_1-\tau_2)^{-\alpha-1} d\tau_2 d\tau_1$$

where t, s ≥ 0 and $f(t) = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(a\alpha t)$.

Proof. Let $0 < \alpha < 1$, $\beta > 1$, $a > 0$. Define

$$f(t) = t^{\beta-1}E^{\gamma}_{\alpha,\beta}(a t^{\alpha}) == \sum_{k=0}^{\infty}\frac{(\gamma)_k t^{k\alpha+\beta-1}}{k!\Gamma(\alpha k+\beta)}$$

then its Caputo derivative satisfies

$$D_t^{\alpha} f(t) = af(t) + \frac{t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}.$$

Now

$$D_t^{\alpha} f(s+t) = af(s+t) + \frac{(s+t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} - \frac{1}{\Gamma(1-\alpha)} \times \int_0^s (s+t-\tau)^{-\alpha}\frac{df(t)}{d\tau}d\tau.$$

or

$$D_t^{\alpha} f(s+t) = af(s+t) + \frac{(s+t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} - \frac{t^{-\alpha}f(s)}{\Gamma(1-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)} \times \int_0^s (s+t-\tau)^{-\alpha-1} f(t)d\tau.$$

Applying the Laplace transform, we get

$$p^{\alpha}\overline{f_s(p)} - p^{\alpha-1}f(s) = a\overline{f_s(p)} + \frac{(s+t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} -- p^{\alpha-1}f(s)$$
$$+\frac{\alpha}{\Gamma(1-\alpha)}\int_0^s (s+t-\tau)^{-\alpha-1} f(\tau)d\tau.$$

which reduces to

$$\Gamma(\beta-\alpha)\,p^{\alpha-\beta}\overline{f_s(p)} = \frac{(s+t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^s (s+t-\tau)^{-\alpha-1} f(\tau)d\tau.$$

It can be written as

$$(p^{\alpha}-a)\,\overline{f_s(p)}$$

$$=\frac{(s+t)^{\beta-\alpha-1}\,p^{\alpha-\beta}}{p^{\alpha}-a}+\frac{\alpha\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)}\int_0^s\frac{p^{\alpha-\beta}}{p^{\alpha}-a}(s+t-\tau)^{-\alpha-1}f(\tau)d\tau.$$

Now we use that

$$\overline{f_s(p)}=\frac{p^{\alpha-\beta}}{p^{\alpha}-a}.$$

Finally, Applying inverse Laplace transform, we arrive at

$$\int_0^t(t-\tau)^{\beta-\alpha-1}f(s+\tau)d\tau=\int_0^t(s+t-\tau)^{\beta-\alpha-1}f(\tau)d\tau+\frac{\alpha\Gamma(\beta-\alpha)}{\Gamma(1-\alpha)}$$

$$\times\int_0^s\int_0^t f(\tau_1)f(\tau_2)(s+t-\tau_1-\tau_2))^{-\alpha-1}f(\tau)d\tau_2 d\tau_1\ .$$

By analytic continuation one can see that this formula remains true when $0<\alpha<1$ and $\beta>\alpha$. Note: If we put β=1 in Theorem 2.1, we arrive at the results given in [15].

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