ORIGINAL PAPER

ANALYTICAL SOLUTIONS FOR MHD FLOW OVER A NONLINEAR STRETCHING SHEET USING EXP-FUNCTION METHOD

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Abstract. *Solutions of nonlinear models are of great importance and their significance has increased a lot. In this article, analytical solutions of nonlinear fluid model representing MHD flow over a nonlinear stretching sheet are obtained via Exp-function method. Computational work and succeeding results re-confirm the efficiency of anticipated algorithm. It is experimental that recommended scheme is highly trustworthy and may be extended to other nonlinear models represented in the form of highly nonlinear differential equations.*

Keywords: *MHD flow, Exp-function method, Analytical Solutions, Maple 18.*

1. INTRODUCTION

Flow between sucking or injecting porous domains is a very chief phenomena. Its biological and industrial applications have paying attention of many scientists towards its learning. Since the pioneering work by Berman [1], researchers from all over the world not only showed their attention in such types of flows but they carried Berman's work to new limits and now we have better understanding of these flows. Further investigation and developments are still welcomed and therefore people are still working on these types of problems. Flow between expanding and contracting vessels on the one hand is a very essential transportation process in many industries while on the other hand it is also a potent simplification to the blood flow model and is responsible for inter-body transportation of food and other minerals. To simulate the flow of blood mathematically many fluid models have been used however [2, 3] showed that the most viable model for this purpose is the Casson fluid model as its rheological properties are very similar to blood and it depicts the shear thinning behavior of blood which cannot be described correctly by Newtonian fluids. Most of the studies done earlier [4-8] refer to the flow of a non-conducting fluid in expanding contracting domains; however, blood and many other fluids in industries contain metallic impurities and are electrically conducting so to understand the flow behavior we need to consider this aspect as well. These conducting fluids under the influence of magnetic field not only behave differently but there is also a visible change in pressure distribution across the flow [9] studied effects of magnetic field on flow of electrically conducting Newtonian fluid flowing between expanding and contracting walls. Motivated by the work mentioned above we present this article to discuss the solitary wave behavior of MHD flow over a nonlinear Stretching sheet. This work can be help in making of better blood flow simulating software;

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which, now a days are used not only in different cardiac tests but also in fabrication of synthetic organs. Many researchers [10-24] adopted the discussed algorithm for finding exact solutions of nonlinear problems. Scientists extend proposed scheme to verdict solitary wave solution of boundary value problems. Analytical solutions of generalized evolution equation are obtained by using mentioned technique and extended for highly nonlinear problems. In his article Ebaid [25] used an approach to calculate values of *c, d, p* and *q* by different way associating highly linear and nonlinear terms.

The elementary inspiration is that we used Exp-function method to deliberated solitary wave performance of the MHD Flow over a nonlinear stretching sheet. The proposed method is very well-matched, highly effective, and awfully consistent for evolution equations and can be prolonged to nonlinear models arising in many fields like engineering, plasma physics and fluid mechanics.

2. FORMULATION OF PROBLEM

Consider fluid flow in a plane with flow along x-direction and deformation perpendicular to flow direction. Magnetic field effects are incorporated and path of magnetic field is along y-direction .The field equations for fluid flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = v\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2(x)}{\rho}u \tag{2}$$

In above u and v are velocity components along x and y axis's respectively and υ, ρ and σ are the kinematics viscosity, fluid density and electrical conductivity respectively. The external electrical field and polarization effects are not considered

$$B(x) = B_0 x^{\frac{n-1}{2}}. \tag{3}$$

The subjected boundary conditions for nonlinear flow are

$$u(x,0) = cx^n, \quad v(x,0) = 0, \quad u(x,y) \to 0, \ \ y \to \infty. \tag{4}$$

Using similar transformation

$$\eta = \sqrt{\frac{c(n+1)}{2\nu}}\, x^{\frac{(n-1)}{2}} y, \qquad u = cx^n f'(\eta),$$
$$v = -\sqrt{\frac{c(n+1)}{2\nu}}\, x^{\frac{(n-1)}{2}} \left[f(\eta) + \frac{n-1}{n+1}\eta f'(\eta)\right], \tag{5}$$

Applying transformation on field equations and boundary conditions we obtain

$$f''' + ff'' - \beta(f')^2 - Mf' = 0 \tag{6}$$

$$f(0)=0, f'(0)=1, f'(\infty)=0, \tag{7}$$

In equation (6)

$$\beta=\frac{2n}{1+n}, M=\frac{2\sigma B_0^2}{\sigma c(1+n)} \tag{8}$$

3. EXP-FUNCTION TECHNIQUE

Nonlinear ordinary differential equation in the general form is

$$F(\phi,\phi',\phi'',\phi''',\ldots)=0 \tag{9}$$

Where prime represents differentiation w.r.t η

Permitting to Exp-function method, we observed that the solitary wave solutions can be articulated in the subsequent procedure

$$\phi(\eta)=\frac{\sum_{i=-c}^{d} a_i e^{i\eta}}{\sum_{j=-p}^{q} b_j e^{j\eta}} \tag{10}$$

In last equation *c, d* and *p, q* are the positive integers and need to be calculated, a_i and b_j are constants. Equation (10) can be expressed in the subsequent corresponding way

$$\phi(\eta)=\frac{a_c e^{c\eta}+\cdots+a_{-d} e^{-d\eta}}{b_p e^{p\eta}+\cdots+a_{-q} e^{-q\eta}} \tag{11}$$

The outcome of equivalent formulation is an imperative and vital analytic solutions of the governing differential equation. Calculating values by using [25], finally results in

$$p=c, q=d \tag{12}$$

3.1. SOLUTION PROCEDURE

The modeled ordinary differential equation is of the following form

$$f'''+ff''-\beta(f')^2-Mf'=0, \tag{13}$$
$$f(0)=0,\ f'(0)=1\ \ f'(\infty)=0, \tag{14}$$

Where

$$\beta=\frac{2n}{1+n},\quad M=\frac{2\sigma B_0^2}{\sigma C(1+n)} \tag{15}$$

Where the prime symbolizes the differentiation w.r.t η. The solution of the equation (13) can be conveyed in the procedure (11). We will elucidate that the final solution does not powerfully rest on the optimal of values of c and d.

Case. I For easiness, we fixed $p = c = 1$ and $q = d = 1$ equation (11) shrinks to

$$\phi(\eta) = \frac{a_1 e^{\eta} + a_0 + a_{-1} e^{-\eta}}{b_1 e^{\eta} + b_0 + b_{-1} e^{-\eta}} \tag{16}$$

Replacing value from (16) into equation (13), we get

$$\frac{1}{A}\left[c_4 e^{4\eta} + c_3 e^{3\eta} + c_2 e^{2\eta} + c_1 e^{\eta} + c_0 + c_{-1} e^{-\eta} + c_{-2} e^{-2\eta} + c_{-3} e^{-3\eta} + c_{-4} e^{-4\eta}\right] = 0 \tag{17}$$

Where $A = \left(b_1 e^{\eta} + b_0 + b_{-1} e^{-\eta}\right)^4$ and c_k are the constants acquired by Maple 18. Associating the coefficients of $e^{i\eta}$ equal to zero, we gain

$$\left[c_4 = 0, c_3 = 0, c_2 = 0, c_1 = 0, c_0 = 0, c_{-1} = 0, c_{-2} = 0, c_{-3} = 0, c_{-4} = 0\right] \tag{18}$$

The solution sets satisfying the equation (13) are given below

1st Solution set

$$\left\{a_{-1} = -\frac{1}{7}(-5M + 5)b_{-1}, a_0 = -\frac{1}{7}(-5M + 5)b_0, a_1 = -\frac{1}{7}(-5M + 5)b_1, b_{-1} = b_{-1}, b_0 = b_0, b_1 = b_1\right\}$$

We get the subsequent generalized solitary solution $f(\eta)$ of equation (13)

$$f(\eta) = \frac{-\frac{1}{7}(-5M + 5)b_{-1}e^{-\eta} - \frac{1}{7}(-5M + 5)b_1 e^{\eta}}{b_{-1}e^{-\eta}}$$

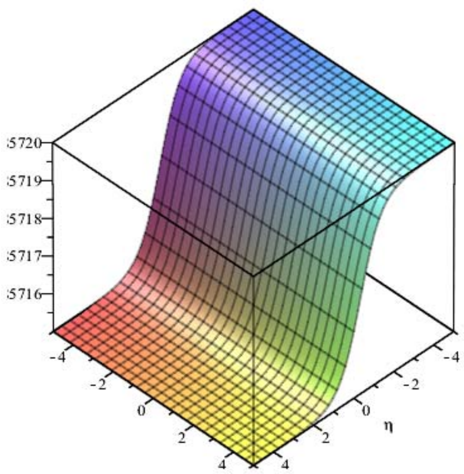

Figure 1. Soliton solutions of equation (13) with $a_1 = 1, b_0 = .1, b_{-1} = 1, b_1 = 2, M = 1.5$.

Case. II If $p = c = 2$ and $q = d = 1$ then trial solution, equation (11) reduces to

$$\phi(\eta)=\frac{a_2 e^{2\eta}+a_1 e^{\eta}+a_0+a_{-1} e^{-\eta}}{b_2 e^{2\eta}+b_1 e^{\eta}+b_0+b_{-1} e^{-\eta}} \qquad (19)$$

Proceeding as before, we have following solution sets satisfy the given equation (13)

1st Solution set

$$\left\{a_{-1}=0, a_0=a_0, a_1=\frac{a_0 b_1}{b_0}, a_2=\frac{a_0 b_2}{b_0}, b_{-1}=0, b_0=b_0, b_1=b_1, b_2=b_2\right\}$$

We acquire the generalized solitary wave solution of equation (13)

$$f(\eta)=\frac{a_0+\dfrac{a_0 b_1 e^{\eta}}{b_0}+\dfrac{a_0 b_2 e^{2\eta}}{b_0}}{b_0+b_1 e^{\eta}+b_2 e^{2\eta}}$$

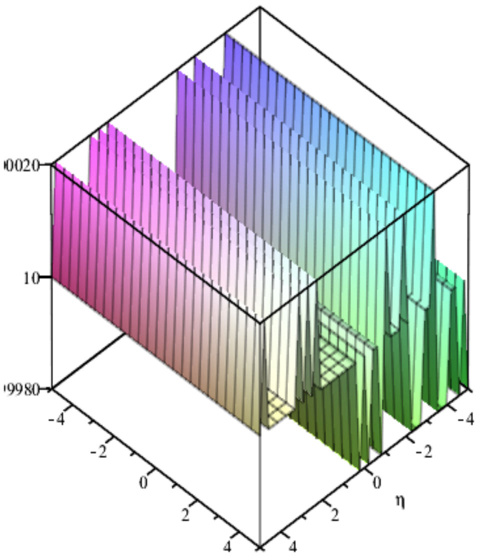

Figure 2. Soliton solutions of equation (13) with $a_0=1, b_2=1, b_{-1}=1, b_1=2, M=1.5$.

2nd solution set

$$\left\{a_{-1}=0, a_0=\frac{1}{2}mb_0-2b_0, a_1=0, a_2=\frac{1}{2}mb_2-2b_2, b_{-1}=0, b_0=b_0, b_1=0, b_2=b_2\right\}$$

Thus obtain the generalized solitary wave solution of equation (13)

$$f(\eta)=\frac{\frac{1}{2}mb_0-2b_0+\left(\frac{1}{2}mb_2-2\right)}{b_0+b_2 e^{2\eta}}$$

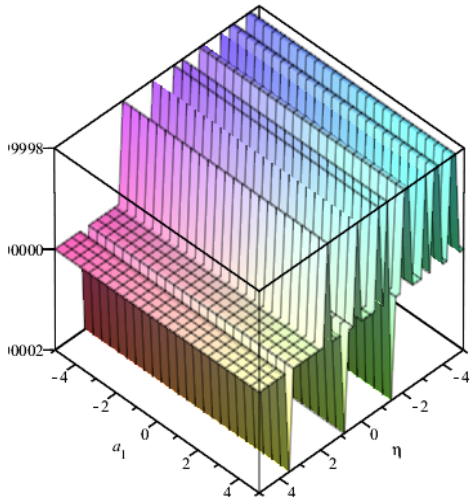

Figure 3. Soliton solutions of equation (13) with $a_0=1, b_2=2, b_{-1}=1, b_0=1, M=1.5$.

3rd solution set

$$\left\{a_{-1}=a_{-1}, a_0=\frac{a_{-1}b_0}{b_{-1}}, a_1=0, a_2=\frac{a_{-1}b_2}{b_{-1}}, b_{-1}=b_{-1}, b_0=b_0, b_1=0, b_2=b_2\right\}$$

Therefore we observed the solitary wave solution of equation (13)

$$f(\eta)=\frac{a_{-1}e^{-\eta}+\dfrac{a_{-1}b_0}{b_{-1}}+\dfrac{a_{-1}b_2e^{2\eta}}{b_{-1}}}{b_{-1}e^{-\eta}+b_0+b_2e^{2\eta}}$$

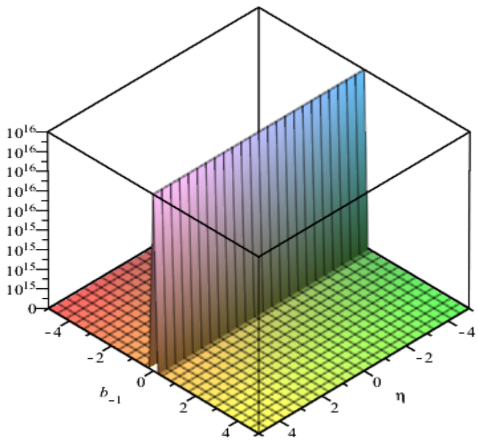

Figure 4. Soliton solutions of equation (13) with $a_{-1}=1, b_1=1, b_{-1}=1, b_0=.1, M=1.5$.

4th solution set

$$\left\{a_{-1}=\frac{a_2b_{-1}}{b_2}, a_0=\frac{a_2b_0}{b_2}, a_1=\frac{a_2b_1}{b_2}, a_2=a_2, b_{-1}=b_{-1}, b_0=b_0, b_1=b_1, b_2=b_2\right\}$$

From now we find the solitary wave solution of equation (13)

$$f(\eta)=\frac{\frac{a_2 b_{-1}e^{-\eta}}{b_2}+\frac{a_2 b_0}{b_2}+\frac{a_2 b_1 e^{\eta}}{b_2}+a_2 e^{2\eta}}{b_{-1}e^{-\eta}+b_0+b_1 e^{\eta}+b_2 e^{2\eta}}$$

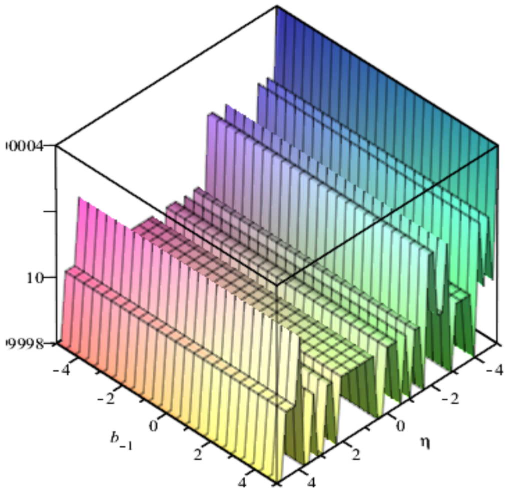

Figure 5. Soliton solutions of equation (13) with $a_2=1, b_0=.1, b_2=.1, b_{-1}=1, b_1=2, M=1.5$.

By computational work and graphical analysis it is observed that for different selections of *c, p, d* and *q* results in various types of soliton solutions. So clearly it is illustrated that final solution does not strongly to be dependent on these parameters.

4. CONCLUSION

In this work, solitary wave solutions for nonlinear fluid model namely MHD flow over a nonlinear stretching sheet are obtained successfully by making use of Exp-function method. The solutions obtained are analytical in nature and are of great significance. Behavior of the solutions obtained using the considered algorithm are discussed with the help of graphs. One important finding is that by using Exp-fucntion method, we can conveniently obtain solitary wave solutions of different nonlinear problems whose solutions cannot be obtained by other classical techniques.

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