ORIGINAL PAPER

THE EVOLUTE SURFACE OF A SURFACE WHICH PARAMETER LINES ARE LINES OF CURVATURE IN E⁴

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Abstract. *In this study it is searched that how the evolute surface of a surface, which is given parameter lines are lines of curvature in $\mathbb{E}^4$ is constructed. Then, the Gaussian curvature and the mean curvature of the evolute surface are investigated and some important results are given. Finally, we present some examples to confirm our claim.*
Keywords: *Normal transport surface, evolute surface, offset functions.*

1. INTRODUCTION

The normal transport surface $\overline{M}$ of M are the generalization of offset surfaces to 4-dimensional Euclidean space E^4 [3]. Observe that, evolute surfaces in E^4 are the special type normal transport surfaces [1, 2, 4, 5]. The paper organized as follows. In section 2, we briefly considered basic concepts of surfaces in Euclidean spaces. In section 3, we consider the evolute surface of a surface which parameter lines are lines of curvature in E^4 and give some important propositions and corollaries. Further we give some examples of evolute surfaces in E^4 .

2. PRELIMINARIES

Let M be a local surface in E^4 given with the regular patch $X(u,v):(u,v)\in D\subset\mathbb{R}^2$. The tangent space $T_M(P)$ to M at an arbitrary point $P=X(u,v)$ of M is spanned by $\{X_u,X_v\}$. Further, X_u and X_v are linearly independent tangential vectors of mapping X at an arbitrary point P and

$$T_M(P)\cong\mathbb{R}^2.$$

In particular, this leads up to decomposition

$$E^4=T_M(P)\oplus T_M^{\perp}(P)$$

with the two dimensional normal space

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$$T_M^{\perp}(P)=\left\{N_\sigma(P)\in E^4 \middle| \left\langle N_\sigma(P),X_u(P)\right\rangle=\left\langle N_\sigma(P),X_v(P)\right\rangle=0,\sigma=1,2\right\}.$$

Definition 1. Let $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ be a surface and the normal space of M is denoted by $T_M^{\perp}(P)=\left\{N_\sigma(P)\in E^4 \middle| \left\langle N_\sigma(P),X_u(P)\right\rangle=\left\langle N_\sigma(P),X_v(P)\right\rangle=0,\sigma=1,2\right\}$.

If along the whole surface $\left\langle N_\sigma(P),N_\upsilon(P)\right\rangle=\delta_{\sigma\upsilon}:=\begin{cases}1, & if\,\sigma=\upsilon\\ 0 & if\,\sigma\neq\upsilon\end{cases}$ is satisfied, the normal space of M is defined by orthonormal normal space of M (or *ONF*) [3].

Definition 2. The *first fundamental form* of $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ is defined by

$$I(X)=\left(g_{ij}\right)_{i,j=1,2},g_{ij}=\left\langle X_{u^i},X_{u^j}\right\rangle$$

setting $u^1:=u$ and $u^2:=v$ [3].

Definition 3. The *second fundamental form* of $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ w.r.t. any unit normal vector $N_\sigma,\sigma=1,2$ is defined by

$$II_{N_\sigma}(X)=\left(L_{\sigma,ij}\right)_{i,j=1,2},L_{\sigma,ij}:=\left\langle X_{u^i},N_{\sigma,u^j}\right\rangle=-\left\langle X_{u^iu^j},N_\sigma\right\rangle$$

setting $u^1:=u$ and $u^2:=v$ [3].

Note that, if the parameter lines of M are lines of curvature, $g_{12}=L_{\sigma,12}=0$.

Definition 4. Let the parameter lines of the surface $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ be lines of curvature. Then, the principal curvatures of M are defined as $k_i^\sigma=\dfrac{L_{\sigma,ii}}{g_{ii}}$ [6].

Definition 5. The *Gaussian curvature* of the surface $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ is defined as [3]:

$$K:=\sum_{\sigma=1}^{2}K_\sigma,K_\sigma:=\frac{L_{\sigma,11}L_{\sigma,22}-L_{\sigma,12}^2}{g_{11}g_{22}-g_{12}^2} \tag{1}$$

Definition 6. The *mean curvature vector* of the surface $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ is defined as [3]:

$$\overrightarrow{H}:=\sum_{\sigma=1}^{2}H_\sigma N_\sigma,H_\sigma:=\frac{L_{\sigma,11}g_{22}-2L_{\sigma,12}g_{12}+L_{\sigma,22}g_{11}}{2\left(g_{11}g_{22}-g_{12}^2\right)}. \tag{2}$$

Definition 7. The *Gaussian torsion* of the surface $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ is defined as [3]:

$$K_N=\frac{1}{W^3}\left[g_{11}\left(L_{1,12}L_{2,22}-L_{2,12}L_{1,22}\right)-g_{12}\left(L_{1,11}L_{1,22}-L_{2,11}L_{1,22}\right)+g_{22}\left(L_{1,11}L_{2,12}-L_{2,11}L_{1,12}\right)\right] \tag{3}$$

Definition 7. Let the surface $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ with an *ONF* $N=\left\{N_1,N_2\right\}$ be given. The torsion coefficients of *ONF* N are defined as [3]:

$$T_{\sigma,i}^{\upsilon}:=\left\langle N_{\sigma,u^i},N_{\upsilon}\right\rangle;i,\sigma,\upsilon=1,2$$

Definition 8. *(Weingarten equations)* Let the surface $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ with an *ONF* $N=\left\{N_1,N_2\right\}$ is given. Then there hold [3]:

$$N_{\sigma,u^i}=-\sum_{j,k=1}^{2}L_{\sigma,ij}g^{jk}X_{u^k}+\sum_{\vartheta=1}^{2}T_{\sigma,i}^{\vartheta}N_{\vartheta}\ .$$

Furthermore, if the parameter lines of M are lines of curvature, Weingarten equations is given as following:

$$\begin{cases}N_{1,u}=k_1^1X_u & N_{1,v}=k_2^1X_v\\ N_{2,u}=k_1^2X_u & N_{2,v}=k_2^2X_v\end{cases}\tag{4}$$

Definition 9. The *curvature tensor* $S_{\sigma,ij}^{\omega}$ of the normal bundle of the surface $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ is defined as

$$\begin{aligned}S_{\sigma,ij}^{\omega}&:=\partial_{u^j}T_{\sigma,i}^{\omega}-\partial_{u^i}T_{\sigma,j}^{\omega}+\sum_{\vartheta=1}^{2}\left(T_{\sigma,i}^{\vartheta}T_{\vartheta,j}^{\omega}-T_{\sigma,j}^{\vartheta}T_{\vartheta,i}^{\omega}\right)\\&=\sum_{m,n=1}^{2}\left(L_{\sigma,im}L_{\omega,jn}-L_{\sigma,jm}L_{\omega,in}\right)g^{mn}\end{aligned}$$

for $i,j,\sigma,\omega=1,2$ [3].

Definition 10. Let the surface $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ is given. Then the *scalar curvature* of its normal bundle is defined as

$$S:=\frac{1}{W}S_{1,12}^{2}$$

where W is the area element of M [3].

Definition 11. Let the surface $M=\left\{X:D\subset\mathbb{R}^2\to E^4\right\}$ with with an *ONF* $N=\left\{N_1,N_2\right\}$ be given. The *normal transport surface* of M is defined as

$$\overline{M}:=\left\{R(u,v)\,\middle|\,R(u,v)=X(u,v)+f(u,v)N_1(u,v)+g(u,v)N_2(u,v)\right\}\tag{5}$$

where $f,g:D\subset\mathbb{R}^2\to\mathbb{R}$ are offset functions [3].

The mapping R is then called an *evolute surface* to X if the tangential planes of R agree with the normal planes of X at corresponding points.

Proposition 1. Let $f, g \neq 0$. If R is an evolute surface to X, then X has flat normal bundle, i.e. it holds $S \equiv 0$ [3].

Theorem 1. If asurface $\overline{M} \subset E^4$ is the evolute for some surface M, then the Levi-Civita connection of $\overline{M}$ is locally flat [2].

Theorem 2. Minimal surfaces have no evolutes [2].

3. EVOLUTE SURFACES IN E4

We would like to point out that, we will assume that the parameter lines of the surface M are lines of curvature along the whole study.

Proposition 2. Let $M \subset E^4$ be a surface and the parameter lines of M be lines of curvature. Then, the normal transport surface $\overline{M}$ of M is the evolute surface of M if and only if

$$\begin{cases} f = \dfrac{k_1^2 - k_2^2}{k_1^1 k_2^2 - k_2^1 k_1^2} \\ g = \dfrac{k_2^1 - k_1^1}{k_1^1 k_2^2 - k_2^1 k_1^2} \end{cases}$$

where f and g are continuous offset functions and k_i^j are principal curvatures of M, $i, j = 1, 2$.

Proof. From (4) and (5), we have

$$\begin{cases} R_u = \left(1 + f k_1^1 + g k_1^2\right) X_u + f_u N_1 + g_u N_2 \\ R_v = \left(1 + f k_2^1 + g k_2^2\right) X_v + f_v N_1 + g_v N_2 \end{cases} \tag{6}$$

Since $\overline{M}$ is the evolute surface of M, the tangent planes of M and $\overline{M}$ are orthogonal to each other at the corresponding points of the surfaces, i.e., $R_u \perp X_u$, $R_u \perp X_v$, $R_v \perp X_u$ and $R_v \perp X_v$. From hence, $R_u^{\mathrm{T}} = 0$ and $R_v^{\mathrm{T}} = 0$. Namely,

$$\begin{cases} k_1^1 f + k_1^2 g = -1 \\ k_2^1 f + k_2^2 g = -1 \end{cases}.$$

If we solve this system of linear equations using Cramer's rule, we have $\begin{cases} f = \dfrac{k_1^2 - k_2^2}{k_1^1 k_2^2 - k_2^1 k_1^2} \\ g = \dfrac{k_2^1 - k_1^1}{k_1^1 k_2^2 - k_2^1 k_1^2} \end{cases}$ where $\Delta = \begin{vmatrix} k_1^1 & k_1^2 \\ k_2^1 & k_2^2 \end{vmatrix} \neq 0$.

Proposition 3. Let $\overline{M}$ be the normal transport surface of M. Then in view of the regularity asummption of $\overline{M}$,there hold necessary

$$\begin{cases} f_u k_2^1 + g_u k_2^2 = 0 \\ f_v k_1^1 + g_v k_1^2 = 0 \end{cases}.$$

Proof. Using (6) we obtain

$$\begin{cases} R_{uv} = \left(f_u k_2^1 + g_u k_2^2 \right) X_v + f_{uv} N_1 + g_{uv} N_2 \\ R_{vu} = \left(f_v k_1^1 + g_v k_1^2 \right) X_u + f_{vu} N_1 + g_{vu} N_2 \end{cases} \tag{7}$$

Since f and g are continuous functions $f_{uv} = f_{vu}$ and $g_{uv} = g_{vu}$. Also, taking into consideration the regularity assumption of $\overline{M}$, we get

$$\left(f_u k_2^1 + g_u k_2^2 \right) X_v = \left(f_v k_1^1 + g_v k_1^2 \right) X_u \tag{8}$$

Taking the dot product of (8) with X_v and X_u , respectively, we have

$$\begin{cases} \left(f_u k_2^1 + g_u k_2^2 \right) g_{22} = 0 \\ \left(f_v k_1^1 + g_v k_1^2 \right) g_{11} = 0 \end{cases}.$$

It is known that both of g_{11} and g_{22} can not be vanish. So, the proof is completed.

Proposition 4. Let $\overline{M}$ be the evolute surface of M and denote by $\overline{K}$ the Gaussian curvature of $\overline{M}$. Then it holds $\overline{K} = 0$.

Proof. From (6) the coefficients of the first fundamental form of $\overline{M}$ are

$$\begin{cases} \overline{g}_{11} = f_u^2 + g_u^2 \\ \overline{g}_{12} = f_u f_v + g_u g_v \\ \overline{g}_{22} = f_v^2 + g_v^2 \end{cases} \tag{9}$$

where $\langle , \rangle$ is the standart scalar product in E^4.

Let us consider an orthonormal normal frame (ONF) of $\overline{M}$. Since $\overline{M}$ is the evolute surface of M

$$\chi(M)\perp\chi\left(\overline{M}\right)\Rightarrow\chi(M)\parallel\chi^{\perp}\left(\overline{M}\right).$$

Because of this reason, we can choose an *ONF* $\overline{N}=\left\{\overline{N}_1,\overline{N}_2\right\}$ of $\overline{M}$, such that

$$\begin{cases}\overline{N}_1=\dfrac{1}{\sqrt{g_{11}}}X_u\\ \overline{N}_2=\dfrac{1}{\sqrt{g_{22}}}X_v\end{cases}\tag{10}$$

Hence, using (7) and (10) we can calculate the coefficients of the second fundamental form of $\overline{M}$ as following:

$$\begin{cases}\overline{L}_{1,11}=-\sqrt{g_{11}}\left(f_u k_1^1+g_u k_1^2\right)\\ \overline{L}_{1,12}=\overline{L}_{1,22}=\overline{L}_{2,11}=\overline{L}_{2,12}=0\\ \overline{L}_{2,22}=-\sqrt{g_{22}}\left(f_v k_2^1+g_v k_2^2\right)\end{cases}\tag{11}$$

Then, from (1) and using (9) and (11) the Gaussian curvatures of $\overline{M}$ w.r.t. the unit normal vectors $\overline{N}_\sigma$, $\sigma=1,2$ can be obtained as follows: $\overline{K}_1=\overline{K}_2=0$.

From herethe Gaussian curvature of $\overline{M}$ is $\overline{K}=\overline{K}_1+\overline{K}_2=0$.

The easy consequences of Proposition 4 are the followings:

Corollary 1. The evolute surface $\overline{M}$ of M is a developable surface.

Corollary 2. Let $\overline{M}$ be the evolute surface of M and denote by $\overline{\omega}$ the area element of $\overline{M}$. Then it holds

$$\overline{\omega}=\left|f_u g_v-g_u f_v\right|.$$

Corollary 3. Let $\overline{M}$ be the evolute surface of M. Then the parameter lines of $\overline{M}$ are lines of curvature if and only if

$$f_u f_v+g_u g_v=0.$$

Corollary 4. Let $\overline{M}$ be the evolute surface of M and denote by $\overline{k}_i^j$, $i,j=1,2$, the principal curvatures of $\overline{M}$. Then it holds

$$\begin{cases}\overline{k}_1^1=-\dfrac{\sqrt{g_{11}}\left(f_u k_1^1+g_u k_1^2\right)}{f_u^2+g_u^2} & \overline{k}_2^1=0\\ \overline{k}_1^2=0 & \overline{k}_2^2=-\dfrac{\sqrt{g_{22}}\left(f_v k_2^1+g_v k_2^2\right)}{f_v^2+g_v^2}\end{cases}.$$

Proposition 5. Let $\overline{M}$ be the evolute surface of M and denote by $\overrightarrow{\overline{H}}$ the mean curvature vector of $\overline{M}$. Then it holds

$$\overline{\overline{H}} = -\frac{\sqrt{g_{11}}\left(f_u k_1^1 + g_u k_1^2\right)\left(f_v^2 + g_v^2\right)N_1 + \sqrt{g_{22}}\left(f_v k_2^1 + g_v k_2^2\right)\left(f_u^2 + g_u^2\right)N_2}{2\left(f_u g_v - g_u f_v\right)^2}.$$

Proof. From (2) the mean curvatures of $\overline{M}$ w.r.t. the unit normal vectors N_σ, $\sigma = 1,2$,can be obtained as the following:

$$\overline{H}_1 = -\frac{\sqrt{g_{11}}\left(f_u k_1^1 + g_u k_1^2\right)\left(f_v^2 + g_v^2\right)}{2\left(f_u g_v - g_u f_v\right)^2}$$

and

$$\overline{H}_2 = -\frac{\sqrt{g_{22}}\left(f_v k_2^1 + g_v k_2^2\right)\left(f_u^2 + g_u^2\right)}{2\left(f_u g_v - g_u f_v\right)^2}.$$

Then using (9) and (11) the mean curvature vector $\overline{\overline{H}}$ of $\overline{M}$ is obtained.

Corollary 5. The evolute surface $\overline{M}$ of M is a minimal surface if and only if

$$g_{11}\left(f_u k_1^1 + g_u k_1^2\right)^2\left(f_v^2 + g_v^2\right)^2 + g_{22}\left(f_v k_2^1 + g_v k_2^2\right)^2\left(f_u^2 + g_u^2\right)^2 = 0.$$

Proposition 6. Let $\overline{M}$ be the evolute surface of M and denote by the Gaussian torsion of $\overline{M}$. Then it holds $\overline{K}_N = 0$.

Proof. From (3) and using (9) and (11) we get

$$\overline{K}_N = 0.\left(-\sqrt{g_{22}}\left(f_v k_2^1 + g_v k_2^2\right) - 0\right) + 0.\left(-\sqrt{g_{11}}\left(f_u k_1^1 + g_u k_2^1\right) - 0\right) = 0.$$

Corollary 6. The evolute surface $\overline{M}$ of M has flat normal bundle, i.e. $R_u^\perp = 0$.

Example 1. Let

$$M \ldots X(u,v) = \left(u, v, u^2, v^2\right)$$

be a surface in E^4.

We can give the first and second partial derivatives of M as follows:

$$X_u(u,v) = (1,0,2u,0)$$
$$X_v(u,v) = (0,1,0,2v)$$
$$X_{uu}(u,v) = (0,0,2,0)$$
$$X_{uv}(u,v) = (0,0,2,0)$$
$$X_{vv}(u,v) = (0,0,0,2)$$

and the orthonormal normal space of M is spanned by

$$N_1(u,v) = \frac{1}{\sqrt{1+4u^2}}(-2u,0,1,0)$$
$$N_2(u,v) = \frac{1}{\sqrt{1+4v^2}}(0,-2v,0,1).$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{cases} g_{11} = 1+4u^2 \\ g_{12} = 0 \\ g_{22} = 1+4v^2 \end{cases}.$$

Similarly, the coefficients of the second fundamental form of the surface are

$$\begin{cases} L_{1,11} = -\dfrac{2}{\sqrt{1+4u^2}} \\ L_{1,12} = L_{1,22} = L_{2,11} = L_{2,12} = 0 \\ L_{2,22} = -\dfrac{2}{\sqrt{1+4v^2}} \end{cases}.$$

From here, we can say that the parameter lines of M are lines of curvature. In addition that M is not a minimal surface and has flat normal bundle. Also, the principal curvatures of M are

$$\begin{cases} k_1^1 = -\dfrac{2}{\left(1+4u^2\right)^{3/2}} \\ k_2^1 = k_1^2 = 0 \\ k_2^2 = -\dfrac{2}{\left(1+4v^2\right)^{3/2}} \end{cases}.$$

Thus, offset functions can be calculated as follows:

$$f = \frac{\left(1+4u^2\right)^{3/2}}{2};\ g = \frac{\left(1+4v^2\right)^{3/2}}{2}$$

where $\Delta = \begin{vmatrix} k_1^1 & k_1^2 \\ k_2^1 & k_2^2 \end{vmatrix} \neq 0$. After all, the evolute surface of M can be obtained as

$$\overline{M}\ldots R(u,v) = \left(-4u^3, -4v^3, \frac{1+6u^2}{2}, \frac{1+6v^2}{2}\right).$$

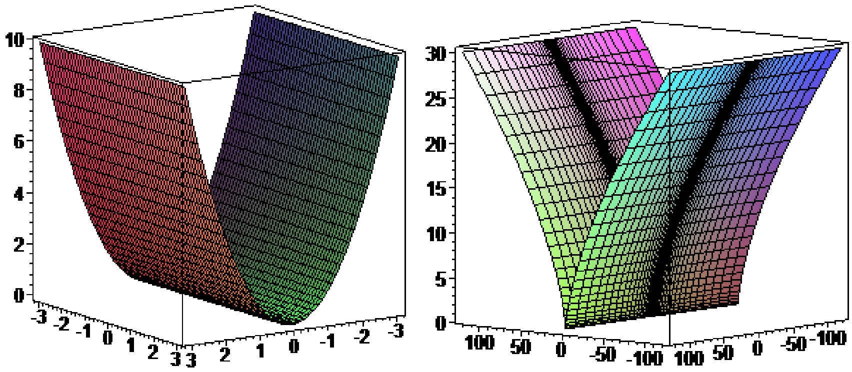

Figure 1. The surface M and its evolute $\overline{M}$.

Example 2. Let $M ... X(u,v) = (u, v, u^2 - v^2, u^2 + v^2)$ be a surface in E^4. We can give the first and second partial derivatives of M as follows:

$$X_u(u,v) = (1,0,2u,2u)$$
$$X_v(u,v) = (0,1,-2v,2v)$$
$$X_{uu}(u,v) = (0,0,2,2)$$
$$X_{uv}(u,v) = (0,0,0,0)$$
$$X_{vv}(u,v) = (0,0,-2,2)$$

and the orthonormal normal space of M is spanned by

$$N_1(u,v) = \frac{2\sqrt{2}|u|}{\sqrt{1+8u^2}}\left(1,0,-\frac{1}{4u},-\frac{1}{4u}\right)$$
$$N_2(u,v) = \frac{2\sqrt{2}|v|}{\sqrt{1+8v^2}}\left(0,1,\frac{1}{4v},-\frac{1}{4v}\right).$$

Hence, the coefficients of the first fundamental form of the surface are

$$\begin{cases} g_{11} = 1+8u^2 \\ g_{12} = 0 \\ g_{22} = 1+8v^2 \end{cases}.$$

Similarly, the coefficients of the second fundamental form of the surface are

$$\begin{cases} L_{1,11} = \pm\frac{2\sqrt{2}}{\sqrt{1+8u^2}} \\ L_{1,12} = L_{1,22} = L_{2,11} = L_{2,12} = 0. \\ L_{2,22} = \pm\frac{2\sqrt{2}}{\sqrt{1+8v^2}} \end{cases}$$

From here, we can say that the parameter lines of M are lines of curvature. In addition that M is not a minimal surface and has flat normal bundle.

Also, the principal curvatures of M are

$$\begin{cases} k_1^1 = \mp \dfrac{2\sqrt{2}}{\left(1+8u^2\right)^{3/2}} \\ k_2^1 = k_1^2 = 0 \\ k_2^2 = \mp \dfrac{2\sqrt{2}}{\left(1+8v^2\right)^{3/2}} \end{cases}.$$

Thus, offset functions can be calculated as follows:

$$f = \pm \frac{\left(1+8u^2\right)^{3/2}}{2\sqrt{2}}; \; g = \pm \frac{\left(1+8v^2\right)^{3/2}}{2\sqrt{2}}$$

where $\Delta = \begin{vmatrix} k_1^1 & k_1^2 \\ k_2^1 & k_2^2 \end{vmatrix} \neq 0$. After all, the evolute surface of M can be obtained as

$$\overline{M} ... R(u,v) = \left(-8u^3, -8v^3, 3u^2 - 3v^2, \frac{1}{2} + 3u^2 + 3v^2 \right).$$

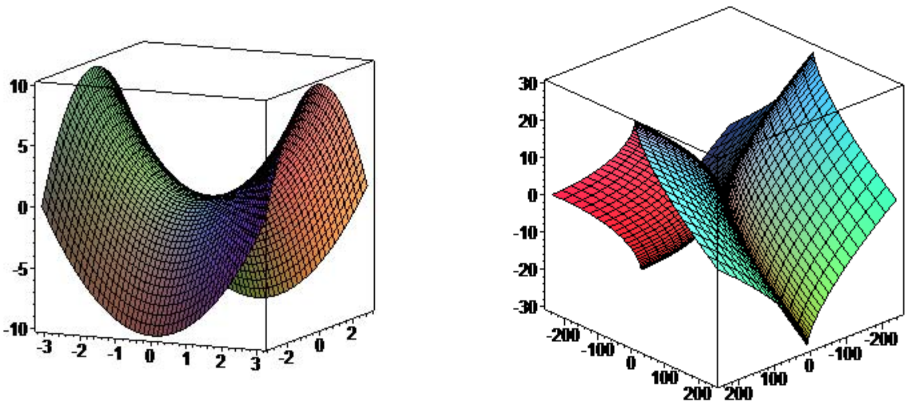

Figure 2. The surface M and its evolute $\overline{M}$.

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