ORIGINAL PAPER

MODULES THAT HAVE A WEAK δ-SUPPLEMENT IN EVERY TORSION EXTENSION

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Manuscript received: 19.01.2017; Accepted paper: 22.04.2017;
Published online: 30.06.2017.

Abstract. *We study modules with the properties $(\delta - TWE)$ and $(\delta - TWEE)$ which are adopted Zöschinger's modules with the properties (E) and (EE). We call a module $(\delta - TWE)$ module if M has a weak δ-supplement in every torsion extension. Similarly if M has ample weak δ-supplements in every torsion extension then M is called $(\delta - TWEE)$ module. We obtain various properties of these modules. We will show that (1) Every direct summand of a $(\delta - TWE)$ module is a $(\delta - TWE)$ module. (2) A module M has the property $(\delta - TWEE)$ iff every submodule of M has the property $(\delta - TWE)$. (3) Any factor module of a $(\delta - TWE)$ module is a $(\delta - TWE)$ module under a special condition. (4) Over a non-local ring, if every submodule of a module M is a $(\delta - TWE)$ module, then it is cofinitely weak δ-supplemented.*

Keywords: *δ-small submodule, weak δ-supplement, torsion extension.*

1. INTRODUCTION

Throughout this paper R will be a commutative domain and all modules are unital left R-modules unless otherwise stated. Let M be an R-module. By $N \leq M$ we mean that N is a submodule of M. Recall that a submodule N of M is called small, denoted by $N \ll M$, if $N + L \neq M$ for all proper submodules L of M. Dually a submodule L of M is said to be essential in M, denoted by $L \trianglelefteq M$, if $L \cap K \neq 0$ for each non zero submodule K of M. [13] A module M is said to be singular if $M \cong N/L$ for some module N and a submodule L of N with $L \trianglelefteq N$ [5].

As a generalization of direct summands of a module one can define supplement submodules. A module M is called supplemented, if every submodule N of M has a supplement in M, i.e. a submodule K of M minimal with respect to $M = N + K$. Equally, K is a supplement of N in M iff $M = N + K$ and $N \cap K \ll K$. If $N + K = M$ and $N \cap K \ll M$, then K is called a weak supplement of N in M. M is weakly supplemented module if every submodule of M has a weak supplement in M. A submodule N of a module M has ample (weak) supplements in M if for all $K \leq M$ with $M = N + K$, there is a (weak) supplement K' of N with $K' \leq K$. If every submodule of M has ample (weak) supplements in M, then M is called amply (weak) supplemented [13].

The concept of δ-small submodules was introduced by Zhou in [14], as a generalization of small submodules. A submodule $N \leq M$ is said to be δ-small in M if $N + X \neq M$ for all proper $X \lneqq M$ with M/X singular. The sum of all δ-small submodules of a module M is denoted by $\delta(M)$. Let K, N be submodules of a module $M. N$ is called a δ-

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supplement of K in M, if $M = N + K$ and $N \cap K \ll_\delta N$ [7]. Similarly N is called a weak δ-supplement of K in M, if $M = N + K$ and $N \cap K \ll_\delta M$ [11]. A module M is called (weak) δ-supplemented if every submodule of M has a (weak) δ-supplement in M. On the other hand, a submodule N of M is said to have ample (weak) δ-supplements in M if every submodule L of M with $M = N + L$ contains a (weak) δ-supplement of N in M. The module M is called amply (weak) δ-supplemented if every submodule of M has ample (weak) δ-supplement in M [11, 12].

Let R be a commutative domain and M be an R-module. We denote by $T(M)$ the set of all elements m of M for which there exists a nonzero element r of R such that $rm = 0$ i.e. $Ann(m) \neq 0$. Then $T(M)$, which is a submodule of M, called the torsion submodule of M. Especially M is called torsion module provided that $T(M) = M$ [13].

For modules $M \leq N$ over commutative domain, we say that N is a torsion extension of M if N/M is torsion. Göçer and Türkmen in [6], studied modules with the property (TE) i.e. modules that have a supplement in every torsion extension. Eryılmaz in [4], studied modules with the property $(\delta - TE)$. Motivated by these we introduce $(\delta - TWE)$ modules i.e. modules that have a weak δ-supplement in every torsion extension. In this study we obtain various properties of modules with the property $(\delta - TWE)$. We show that a class of $(\delta - TWE)$ modules is closed under direct summands and factor modules by a special condition.We prove that every submodule of a module is a $(\delta - TWE)$ module iff it has ample weak δ-supplements in evry torsion extension. We also show that over a non local ring if every submodule of a module M is a $(\delta - TWE)$ module then it is cofinitely weak δ-supplemented.

2. PRELIMINARIES

We will give following lemmas for the completeness.

Lemma 1: Let M be an R-module, then the following statements are equivalent:
1. M is cofinitely δ-supplemented
2. Every maximal submodule of M has a δ-supplement in M [1].

Lemma 2: Let R be a ring which is not local. If M is a simple module then it is torsion [6] .

3. RESULTS AND DISCUSSION

Proposition 1: δ-Hollow modules have the property $(\delta - TWE)$.

Proof: Let S be a δ-hollow module and N be any torsion extension of S. If S is δ-small in N, N is a weak δ-supplement of S in N. Suppose that S is not δ-small in M. Then there is a proper submodule S' of N such that $S + S' = N$ and N/S is singular. If S is simple $S \cap S' = 0$ and so S' is a direct summand of N. In opposite situation since S is δ-hollow, $S \cap S'$ is δ-small in S. In both cases, S' is a weak δ-supplement of S in N.

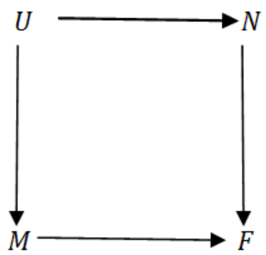
Proposition 2: Every direct summand of a $(\delta - TWE)$ module is a $(\delta - TWE)$ module.

Proof: Let M be a $(\delta - TWE)$ module, U be a direct summand of M and let N be any torsion extension of U. Then $M = A \oplus U$ for some submodule $A \leq M$. We denote by T the external direct sum $A \oplus N$ and consider the canonical embedding $\varphi: M \longrightarrow T$. Then $M \cong \varphi(M)$ is a $(\delta - TWE)$ module and we have $T/\varphi(M) = (A \oplus N)/\varphi(M) \cong (A \oplus N)/(A \oplus U) \cong N/U$ is torsion. Since $\varphi(M)$ is a $(\delta - TWE)$ module, $\varphi(M)$ has a weak δ-supplement V in T, that is, $\varphi(M) + V = T$ and $\varphi(M) \cap V \ll_\delta T$. Fort he projection $\pi: T \longrightarrow N$, we have that $N = U + \pi(V)$. Since $Ker(\pi) \subseteq \varphi(M)$, we get $\pi(\varphi(M) \cap V) \subseteq \pi(\varphi(M)) \cap \pi(V) = U \cap \pi(V) \ll_\delta \pi(T) \leq N$ and so $U \cap \pi(V) \ll_\delta N$ is obtained. Hence $\pi(V)$ is a weak δ-supplement of U in N.

Proposition 3: Let M be a module. Then the following statements are equivalent:
(1) Every submodule of M is a $(\delta - TWE)$ module.
(2) M has ample δ-supplements in every torsion extension i.e. M is a $(\delta - TWEE)$ module.

Proof: $(1) \Longrightarrow (2)$: Suppose that every submodule of M is a $(\delta - TWE)$ module. For a torsion extension $N$of M, let $N = M + K$ for some submodule K of N. Note that $N/M = (M + K)/M \cong K/(M \cap K)$ is torsion. By hypothesis $M \cap K$ is a $(\delta - TWE)$ module and so there exists a submodule L of K such that $K = (M \cap K) + L$ and $(M \cap K) \cap L = M \cap L \ll_\delta K$. Then we have $M \cap L \ll_\delta N$ and $N = M + K = M + (M \cap K) + L = M + L$. Hence L is a weak δ-supplement of M in N.

$(2) \Longrightarrow (1)$: Let M be a module with the property $(\delta - TWEE)$ and let U be any submodule of M. For a cofinite extesion N of U, let $F = (M \oplus N)/H$ where the submodule H is the set of all elements $(a, -a)$ of F with $a \in U$ and let $\alpha: M \longrightarrow F$ via $\alpha(m) = (m, 0) + H$, $\beta: N \longrightarrow F$ via $\beta(n) = (0, n) + H$ for all $m \in M, n \in N$. It is clear that α and β are monomorphisms. Hence we have the following pushout diagram:



where $\mu_1: U \longrightarrow N$ and$\mu_1: U \longrightarrow M$ are inclusion mappings. It is easy to prove that $F = Im(\alpha) + Im(\beta)$. Now we define $\gamma: F \longrightarrow N/U$ by $\gamma((m, n) + H) = n + U$ for all $(m, n) + H \in F$. Then γ is an epimorphism. Note that $Ker(\gamma) = Im(\alpha)$ and so $N/U \cong F/Im(\alpha)$ is finitely generated. Since α is a monomorphism, by assumption, $Im(\alpha)$ has the property $(\delta - TWEE)$. Then it follows immediately that $Im(\alpha)$ has a weak δ-supplement Vin F with $V \leq Im(\beta)$, i.e. $F = Im(\alpha) + V$ and $Im(\alpha) \cap V \ll_\delta F$. Then $N = \beta^{-1}(Im(\alpha)) + \beta^{-1}(V) = U + \beta^{-1}(V)$. Suppose that $U \cap \beta^{-1}(V) + X = N$ for some submodule X of N with N/X singular. Then we have $\beta((U \cap \beta^{-1}(V) + X) = \beta(U \cap \beta^{-1}(V)) + \beta(X) = Im(\alpha) \cap V + \beta(X) = \beta(N)$ since β is a monomorphism. And it is clear that $Im(\alpha) \cap V + \beta(X) + Im(\alpha) = F$. Now we define $\theta: N/X \longrightarrow \beta(N)/\beta(X)$ by $\theta(n + X) = \beta(n) + \beta(X)$ for all $n + X \in N/X$. Note that θ is an isomorphism. Hence $N/X \cong \beta(N)/\beta(X) \cong \beta(N) + Im(\alpha)/\beta(X) + Im(\alpha) = F/(\beta(X) + Im(\alpha))$ is singular. Since $Im(\alpha) \cap V \ll_\delta F$, it follows that $\beta(X) + Im(\alpha) = \beta(N) + Im(\alpha)$. Hence $\beta(X) = \beta(N)$ is obtained because of definition of β. And we have that $X = N$ since β is a monomorphism. That means $U \cap \beta^{-1}(V) \ll_\delta N$. So $\beta^{-1}(V)$ is a weak δ-supplement of U in N.

Proposition 4: Let R be aring which is not local and let M be an R-module. If every submodule of M is a $(\delta - TE)$ module, then it is cofinitely δ-supplemented.

Proof: By [1, Theo. 2.9], it is sufficies to show that every maximal submodule of M has a δ-supplement in M. Let U be any maximal submodule of M. Then, M/U is simple, and so it is torsion by Lemma 1. By the hypothesis U has a δ-supplement in M. Thus M is cofinitely δ-supplemented.

Definition 5: We call a module M is cofinitely weak δ-supplemented module (or briefly δ-cws module) if every cofinite submodule has a weak δ-supplement in M.

Clearly cofinitely δ-supplemented modules and weakly δ-supplemented modules are cofinitely weak δ-supplemented and a finitely generated module is weakly δ-supplemented if and only if it is a δ-cws module.

Lemma 6: Let U and K be submodules of N such that K is a weak δ-supplement of a maximal submodule M of N. If $K + U$ has a weak δ-supplement in N, then U has a weak δ-supplement in N.

Proof: Let X be a weak δ-supplement of $K + U$ in N. If $K \cap (X + U) \leq K \cap M \ll_\delta N$ then $X + K$ is a weak δ-supplement of U since $U \cap (X + K) \leq X \cap (K + U) + K \cap (X + U) \ll_\delta N$. Now suppose that $K \cap (X + U) \nleq K \cap M$. Since $K/(K \cap M) \cong (K + M)/M = N/M$, $K \cap M$ is a maximal submodule of K. Therefore $(K \cap M) + [K \cap (X + U)] = K$. Then X is a weak δ-supplement of U in N since $U \cap X \leq (K + U) \cap X \ll_\delta N$ and $N = X + U + K = X + U + (K \cap M) + [K \cap (X + U)] = X + U$ as $K \cap (X + U) \leq X + U$ and $K \cap M \ll_\delta N$. So in both cases there is a weak δ-supplement of U in N.

For a module N, let Γ be the set of all submodules K such that K is a weak δ-supplement for some maximal submodule of N and let $\delta\text{-}cws(N)$ denote the sum of all submodules from Γ.

Theorem 7: For a module N, the following statements are equivalent:

1. N is a $\delta\text{-}cws$ module;
2. Every maximal submodule of N has a weak δ-supplement;
3. $N/\delta\text{-}cws(N)$ has no maximal submodules.

Proof: $(1) \Longrightarrow (2)$ is obvious since every maximal submodule is cofinite.

$(2) \Longrightarrow (3)$ Suppose that there is a maximal submodule $M/\delta\text{-}cws(N)$ of $N/\delta\text{-}cws(N)$. Then M is a maximal submodule of N. By (2), there is a weak δ-supplement K of M in N. Then $K \in \Gamma$, therefore $K \leq \delta\text{-}cws(N) \leq M$. Hence $N = M + K = M$. This contradiction shows that $N/\delta\text{-}cws(N)$ has no maximal submodules.

$(2) \Longrightarrow (3)$ Let U be a cofinite submodule of N. Then $U + \delta\text{-}cws(N)$ is also cofinite. If $N/[U + \delta\text{-}cws(N)] \neq 0$, by Theorem 2.8 of Anderson and Fuller (1992), there is a maximal submodule $M/[U + \delta\text{-}cws(N)]$ of the finitely generated module $N/[U + \delta\text{-}cws(N)]$. It follows that M is a maximal submodule of N and $M/\delta\text{-}cws(N)$ is a maximal submodule of $N/\delta\text{-}cws(N)$. This conradicts (3). So $N = U + \delta\text{-}cws(N)$. Now, N/U is finitely generated, say by elements $x_1 + U, x_2 + U, \ldots, x_m + U$ therefore $N = U + Rx_1 + Rx_2 + \cdots + Rx_m$. Each element x_i $(i = 1, 2, \ldots, m)$ can be written as $x_i = u_i + c_i$, where

$u_i \in U, c_i \in \delta\text{-}cws(N)$. Since each c_i is contained in the sum of finite number of submodules from Γ, $N = U + K_1 + K_2 + \cdots + K_n$ for some submodules $K_1, K_2, \ldots, K_n$ of N from Γ. Now $N = (U + K_1 + K_2 + \cdots + K_{n-1}) + K_n$ has a weak δ-supplement, namely 0. By Lemma 6 $U + K_1 + K_2 + \cdots + K_{n-1}$ has a weak δ-supplement. Continuing in this way (applying Lemma 6 n-times) we obtain that U has a weak δ-supplement in N.

Lemma 8: Let R be a ring which is not local and let M be an R-module. If every submodule of M is $(\delta - TWE)$ module then it is cofinitely weak δ-supplemented.

Proof: It sufficies to show that every maximal submodule of M has a weak δ-supplement in M. Let U be any maximal submodule of M. Then M/U is simple and so it is torsion by Lemma 5. By the hypothesis U has a weak δ-supplement in M. Thus M is cofinitely weak δ-supplemented.

Proposition 9: Let M be a module and U be a submodule of M. If $U \ll_\delta M$ and the factor module M/U has the property $(\delta - TWE)$, then M also has the property $(\delta - TWE)$.

Proof: Let N be any torsion extension of M. Then we obtain that $N/M \cong (N/U)/(M/U)$ is torsion. Since M/U has the property $(\delta - TWE)$, there exists a submodule V/U of N/U such that $M/U + V/U = N/U$ and $(M/U) \cap (V/U) = (M \cap V)/U \ll_\delta N/U$. Note that $M + V = N$. Suppose that $(M \cap V) + T = N$ for a submodule T of N such that N/T is singular. Then we obtain $(M \cap V)/U + (T + U)/U = N/U$. Since $(M \cap V)/U \ll_\delta N/U$ and $N/T + N \leq N/T$ singular we have that$(T + U)/U = N/U$. It is clear that $T + U = N$. By hypothesis and since N/T is singular it follows that $N = T$. Hence $M \cap V \ll_\delta N$.

Corollary 10: Let M be a finitely generated module. If $M/\delta(M)$ has the property $(\delta - TWE)$, then so does M.

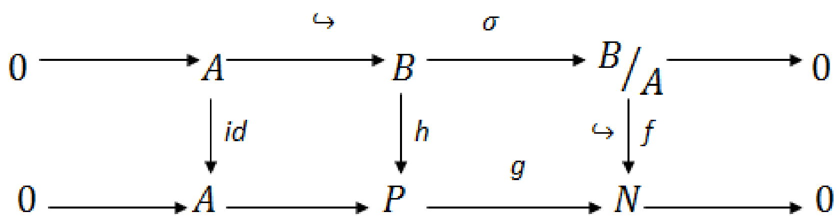
Lemma 11: Let M be a $(\delta - TWE)$ module and N be a torsion extension of M such that $\delta(N) = 0$. Then M is a direct summand of N.

Proof: By assumption, M has a weak δ-supplement in N. Since $M \cap K \ll_\delta K$, it follows from $M \cap V \leq \delta(K) \leq \delta(N) = 0$. Hence $N = M \oplus K$.

Corollary 12: Let M be a $(\delta - TWE)$ module over δ-V-ring. Then M is a direct summand of any module N with N/M torsion.

Theorem 13: Let $A \leq B \leq C$ with (C/A) injective. If B has the property $(\delta - TWE)$, so does B/A.

Proof: Let N be any extension of B/A. So we have the following commutative diagram with exact rows since C/A is injective [by 16, Lemma 2.16].



Since h is monic, $N\ /\ (B\ /\ A) \cong g(N)\ /\ g(B\ /\ A) \cong \sigma^{-1}(g(N))/\sigma^{-1}(g(B/A)\,) = \sigma^{-1}(g(N))\ /\ \sigma^{-1}(\sigma(B)) \cong \sigma^{-1}(g(N))\ /\ B$ is torsion and B has the property $(\delta - TWE)$, $B \cong h(B)$ has a weak δ-supplement V in P (, that is, $h(B) + V = P$ and $h(B) \cap V \ll_{\delta} P$. We claim that $g(V)$ is a weak δ-supplement of B/A in N.

$$B/A + g(V) = (f\sigma)(B) + g(V) = g(h(B)) + g(V) = g(P) = N,\ B/A \cap g(V) = f(\sigma(B)) \cap g(V) = g[h(B) \cap g(V)]\ \ll_{\delta} g(V)$$

since $h(B) \cap V \ll_{\delta} V$ and g is a homomorphism. Hence $B/A \cap g(V) \ll_{\delta} N$.

Corollary 14: Let R be a hereditary ring. If an injective R-module M has the property $(\delta - TWE)$, then so does every factor module of M.

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