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NUMERICAL TREATMENT OF FRACTIONAL ENDEMIC DISEASE MODEL VIA LAPLACE ADOMIAN DECOMPOSITION METHOD

KAMAL SHAH[1], SAMIA BUSHNAQ[2]

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Abstract. *This article is concerned with the approximate solution of fractional order endemic model of non-fatal disease in a community. We consider that a population in the concerned community is initially in equilibrium with an endemic disease caused by a wild type virus. Using Laplace transform coupled with Adomian decomposition method, we obtain numerical solution of the proposed model. The mentioned method is known as Laplace Adomian decomposition method (LADM). The solutions obtained by this method are compared with the solutions obtained by the RK4 and homotopy perturbation method for taking classical order derivative of the governing equations.*

Keywords: *Fractional derivative, endemic model, Laplace Transform, Adomian decomposition method; Analytical solutions.*

1. INTRODUCTION

Mathematical models of the infectious diseases are the important tools to study the mechanism through which diseases spread in a community. They are using for the predictions of future courses, an outbreak and to evaluating strategies to control an epidemic. The earliest idea of mathematical modeling of spread of disease was point out by D. Bernoulli in 1766, which gave birth to the start of modern theoretical epidemiology. W. Hamer and R. Ross are also considered earlier pioneer of modeling of infectious diseases in beginning of 20th century. They used the law of mass action to explain the behavior of epidemic. Later on L. Reed and W.H. Frost established a famous model known as the Reed-Frost epidemic model, which describes the relationship between the susceptible, infected and immune individual in a community. Mckendrick and Kermack [1], formulated a simple deterministic model in 1927 known as SIR model given by

$$\begin{cases} \dfrac{du(t)}{dt} = -\beta u(t)v(t), \\ \dfrac{dv(t)}{dt} = \beta u(t)v(t) - \gamma v(t), \\ \dfrac{dw(t)}{dt} = \gamma v(t) \end{cases} \qquad (1)$$

[1] University of Malakand, Chakadara Dir(L), Department of Mathematics, Khyber Pakhtunkhwa, Pakistan. E-mail: kamalshah408@gmail.com.
[2] Princess Sumaya University for Technology, Department of Basic Sciences, King Abdullah II Faculty of Engineering, Amman 11941, Jordan. E-mail: s.bushnaq@psut.edu.jo.

where the fixed population is consisted of three types which are susceptible, infected and isolated individuals which cannot get or transmit the disease for various preseason. β, γ are the transmission rates between the compartments.

Let α be the birth rate and d the disease unrelated death rate and c is the disease related death rate, $w$is immune class. Then the modified form of the model (1) is obtained in [2], which is given by

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha N - \beta u(t)v(t) - du(t), \\ \dfrac{dv(t)}{dt} = \beta u(t)v(t) - (\gamma + d + c)v(t), \\ \dfrac{dw(t)}{dt} = \gamma v(t) - dw(t) \end{cases} \tag{2}$$

where $N = u_0 + v_0 + w_0$ is not necessarily constant. The constant presence of a disease in a community is called endemic. For example, Malaria is endemic in Sub-Saharan Africa, where 90 percent deaths occur due to malaria. In the presence of two infected individuals $v, \acute{v}$, a single recovered class w and single susceptible class u, the following model as in [3], was established

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha N - du(t) - \beta u(t)v(t) - \acute{\beta} u(t)\acute{v}(t), \\ \dfrac{dv(t)}{dt} = \beta u(t)v(t) - (\gamma + d + c)v(t), \\ \dfrac{d\acute{v}(t)}{dt} = \beta u(t)\acute{v}(t) \acute{-} (\acute{\gamma} + d + \acute{c})\acute{v}(t), \\ \dfrac{dw(t)}{dt} = \gamma v(t) + \acute{\gamma}\acute{v}(t) - dw(t) \end{cases} \tag{3}$$

where β is infectious rate, γ is the removal rate and c is the disease death related rate. While $\acute{\beta}, \acute{\gamma}, \acute{c}$ are the corresponding rates of mutant virus.

These models were studied for their local and global stability in view of classical order derivative, see [2]. The above classical model (1), was solved by Biazer [17], with the help of Adomian decomposition method (ADM), Rafei et al. [18, 19], by mean of homotopoy perturbation method (HPM) and variation iteration method (VIM) respectively. Similarly the same model (1) was solved by Fadi et al. [20], by using homotopy analysis method (HAM) and Abdul Monim et al. [21], by differential transform method (DTM).

In last few decades, it has been found that the area involving fractional order differential equations have significant applications in various disciplines of science and technology; we refer few of them in [4, 6, 13, 17, 20, 22]. The aforementioned models (1), and (2) were also studied by considering their fractional order extension, for details see [23-25]. Therefore, in recent years, the fractional order models were given much attention, because the biological models that involved fractional order derivative are more realistic and accurate as compared to the classical order models, for detail see [7-10]. Motivated by the above work, in this manuscript, we considered the given modified form of model (3) by taking the derivative of the governing equations in fractional order. The arbitrary order shows the realistic biphasic decline behavior of infection of disease with a slower rate. Therefore the modified model of arbitrary order is

$$\begin{cases} {}_{+0}^{\ c}D^{\alpha_1}u(t) = \alpha N - du(t) - \beta u(t)v(t) - \acute{\beta}u(t)\acute{v}(t), \\ {}_{+0}^{\ c}D^{\alpha_2}v(t) = \beta u(t)v(t) - (\gamma + d + c)v(t), \\ {}_{+0}^{\ c}D^{\alpha_3}\acute{v}(t) = \beta u(t)\acute{v}(t) - (\acute{\gamma} + d + \acute{c})\acute{v}(t), \\ {}_{+0}^{\ c}D^{\alpha_4}w(t) = \gamma v(t) + \acute{\gamma}\acute{v}(t) - dw(t), \end{cases} \tag{4}$$

Subject to the initial conditions:

$$u(0) = N_1, v(0) = N_2, \acute{v}(0) = N_3, w(0) = N_4, \tag{5}$$

where $0 < \alpha_i \leq 1$, for $i = 1,2,3,4$.

At $0 < \alpha_i \leq 1$, for $i = 1,2,3,4$, the model(4) will be reduced to classical order model. Here the initial conditions are interdependent on each other and satisfy the relation $N = N_1 + N_2 + N_3 + N_4$, where N is total number of the individuals in the population.

The numerical solutions are well studied for classical order model, however for non-integer order model the numerical solutions are rarely studied by using Adomian decomposition method coupled with integral transform like Laplace transform, Sumudu transform, etc. Therefore, in this article, we develop a simple and easy technique for the numerical solutions to the model (4).

The concerned technique is better than other technique like homotopy analysis and perturbation method and variation method. Because our proposed method provide accurate solutions and its implementation is also easy. We have also compared our solutions with that of the solution obtained by RK4 and homotopy perturbation method. For the developed procedure, we assigned random values to the initial conditions and parameters involved in the model to verify only the established results.

2. NECESSARY FUNDAMENTAL RESULTS AND NOTIONS

Definition 2.1 The Riemann-Liouville fractional integral of a function $F \in L^1([0, b], R)$ of order $q \in R_+$ is given as

$$J^q F(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} F(s)\ ds,$$

Provided that the integral on the right is converges and pointwise defined on $(0, \infty)$.

Definition 2.2 The arbitrary order derivative of a function y in Caputo sense over the interval $[0, b]$ is provided by

$${}_{+0}^{\ c}D^q F(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} F^{(n)}(s)\ ds, \text{ where } n-1 < q \leq n,$$

Provided the integral on the right is pointwise on $(0, \infty)$. While $n = [q] + 1$ and $[q]$ represents the integer part of q. In particularly, if $0 < q < 1$, then

$${}_{+0}^{\ c}D^q F(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{F\prime(s)}{(t-s)^q}\ ds, \text{ where } n-1 < q \leq n,$$

The following result holds for fractional differential equations

$$ {}_{+0}^{c}D^{q}F(t)=0,\ \ n-1<q\leq n,\ \ n=[q]+1, $$
$$ J^{q}[{}_{+0}^{c}D^{q}F(t)](t)=F(t)+\ C_0+\ C_1t+\ C_2t^2+\cdots+\ C_{n-1}t^{n-1}. $$

Note: We use Caputo fractional derivative throughout in this paper as it treat initial value problems of fractional differential equations like classical order differential equations.

Definition 2.3 We recall the definition of Laplace transform of Caputo derivative as:

$$ \mathcal{L}\{{}_{+0}^{c}D^{q}f(t)\}=\ s^{q}F(s)-\sum_{k=0}^{n-1}s^{q-k-1}f^{(k)}(0),\qquad n-1<q\leq n;\ \ n\ \in N. $$

In this section, we develop the general procedure of the considered model (4) together with initial conditions (5). Applying Laplace transform on both sides of the model (4) as

$$ \begin{cases}\mathcal{L}\{{}_{+0}^{c}D^{\alpha_1}u(t)\}=\mathcal{L}\{\alpha N-du(t)-\beta u(t)v(t)-\acute{\beta}u(t)\acute{v}(t)\},\\ \mathcal{L}\{{}_{+0}^{c}D^{\alpha_2}v(t)\}=\mathcal{L}\{\beta u(t)v(t)-(\gamma+d+c)v(t)\},\\ \mathcal{L}\{{}_{+0}^{c}D^{\alpha_3}\acute{v}(t)\}=\mathcal{L}\{\beta u(t)\acute{v}(t)-(\acute{\gamma}+d+\acute{c})\acute{v}(t)\},\\ \mathcal{L}\{{}_{+0}^{c}D^{\alpha_4}w(t)\}=\mathcal{L}\{\gamma v(t)+\ \acute{\gamma}\acute{v}(t)-dw(t)\}.\end{cases} \tag{6} $$

which implies that

$$ \begin{cases}s^{\alpha_1}\mathcal{L}\{u(t)\}-\ s^{\alpha_1-1}u(0)=\mathcal{L}\{\alpha N-du(t)-\beta u(t)v(t)-\acute{\beta}u(t)\acute{v}(t)\},\\ s^{\alpha_2}\mathcal{L}\{v(t)\}-\ s^{\alpha_2-1}v(0)=\mathcal{L}\{\beta u(t)v(t)-(\gamma+d+c)v(t)\},\\ s^{\alpha_3}\mathcal{L}\{\acute{v}(t)\}-\ s^{\alpha_3-1}\acute{v}(0)=\ \mathcal{L}\{\beta u(t)\acute{v}(t)-(\acute{\gamma}+d+\acute{c})\acute{v}(t)\},\\ s^{\alpha_4}\mathcal{L}\{w(t)\}-\ s^{\alpha_4-1}w(0)=\ \mathcal{L}\{\gamma v(t)+\ \acute{\gamma}\acute{v}(t)-dw(t)\}.\end{cases} \tag{7} $$

Using the initial conditions and taking the inverse Laplace transform in system (7), we have

$$ \begin{cases}\mathcal{L}\{u(t)\}=\dfrac{N_1}{s}+\dfrac{1}{s^{\alpha_1}}\mathcal{L}\{\alpha N-du(t)-\beta u(t)v(t)-\acute{\beta}u(t)\acute{v}(t)\},\\ \mathcal{L}\{v(t)\}=\dfrac{N_2}{s}+\ \dfrac{1}{s^{\alpha_2}}\ \mathcal{L}\{\beta u(t)v(t)-(\gamma+d+c)v(t)\},\\ \mathcal{L}\{\acute{v}(t)\}=\ \dfrac{N_3}{s}+\ \dfrac{1}{s^{\alpha_3}}\mathcal{L}\{\beta u(t)\acute{v}(t)-(\acute{\gamma}+d+\acute{c})\acute{v}(t)\},\\ \mathcal{L}\{w(t)\}]=\ \dfrac{N_4}{s}+\ \dfrac{1}{s^{\alpha_4}}\mathcal{L}\{\gamma v(t)+\ \acute{\gamma}\acute{v}(t)-dw(t)\}.\end{cases} \tag{8} $$

Assuming the solutions $u(t), v(t), \acute{v}(t), w(t)$ in the form of infinite series provided by

$$ u=\sum_{n=0}^{\infty}u_n\,, v=\sum_{n=0}^{\infty}v_n\,, \acute{v}=\sum_{n=0}^{\infty}\acute{v}_n\ \ , w=\sum_{n=0}^{\infty}w_n, \tag{9} $$

and the nonlinear terms involved in the model are $u(t)v(t)$ and $u(t)\acute{v}(t)$ are decomposed in term of Adomain polynomials as

$$u(t)v(t)=\sum_{n=0}^{\infty}P_n\,,u(t)\acute{v}(t)=\sum_{n=0}^{\infty}Q_n. \quad (10)$$

where P_n, Q_n are Adomian polynomials defined as

$$P_n=\frac{1}{\Gamma(n+1)}\frac{d^n}{d\eta^n}\left[\sum_{i=0}^{n}\eta^i u_i\,\sum_{i=0}^{n}\eta^i\,v_i\right]\Bigg|_{\eta=0},$$
$$Q_n=\frac{1}{\Gamma(n+1)}\frac{d^n}{d\eta^n}\left[\sum_{i=0}^{n}\eta^i u_i\,\sum_{i=0}^{n}\eta^i\,\acute{v}_i\right]\Big|_{\eta=0}. \quad (11)$$

Substituting (9), (10) in system (8) and equating the corresponding terms on both sides of the equations, we get

$$\begin{cases}\mathcal{L}\{u_0\}=\frac{N_1}{s},\ \mathcal{L}\{v_0\}=\frac{N_2}{s},\ \mathcal{L}\{\acute{v}_0\}=\frac{N_3}{s},\ \mathcal{L}\{w_0\}=\frac{N_4}{s},\\ \mathcal{L}\{u_1\}=\frac{\alpha N}{s^{\alpha_1+1}}-\frac{d}{s^{\alpha_1}}\mathcal{L}\{u_0\}-\frac{\beta}{s^{\alpha_1}}\mathcal{L}\{P_0\}-\frac{\acute{\beta}}{s^{\alpha_1}}\mathcal{L}\{Q_0\},\ \mathcal{L}\{v_1\}=\frac{\beta}{s^{\alpha_2}}\mathcal{L}\{P_0\}-\\ \left(\frac{\gamma+d+c}{s^{\alpha_2}}\right)\mathcal{L}\{v_0\},\ \mathcal{L}\{\acute{v}_1\}=\frac{\acute{\beta}}{s^{\alpha_3}}\mathcal{L}\{Q_0\}-\left(\frac{\acute{\gamma}+d+\acute{c}}{s^{\alpha_3}}\right)\mathcal{L}\{\acute{v}_0\},\ \mathcal{L}\{w_1\}=\frac{\gamma}{s^{\alpha_4}}\mathcal{L}\{v_0\}+\\ \frac{\acute{\gamma}}{s^{\alpha_4}}\mathcal{L}\{w_0\}-\frac{d}{s^{\alpha_4}}\mathcal{L}\{w_0\},\\ \mathcal{L}\{u_2\}=-\frac{d}{s^{\alpha_1}}\mathcal{L}\{u_1\}-\frac{\beta}{s^{\alpha_1}}\mathcal{L}\{P_1\}-\frac{\acute{\beta}}{s^{\alpha_1}}\mathcal{L}\{Q_1\},\mathcal{L}\{v_2\}=\frac{\beta}{s^{\alpha_1}}\mathcal{L}\{P_1\}-\\ \left(\frac{\gamma+d+c}{s^{\alpha_1}}\right)\mathcal{L}\{v_1\},\mathcal{L}\{\acute{v}_2\}=\frac{\acute{\beta}}{s^{\alpha_3}}\mathcal{L}\{vQ_1\}-\left(\frac{\acute{\gamma}+d+\acute{c}}{s^{\alpha_3}}\right)\mathcal{L}\{\acute{v}_1\},\mathcal{L}\{w_2\}=\frac{\gamma}{s^{\alpha_4}}\mathcal{L}\{v_1\}\\ +\frac{\acute{\gamma}}{s^{\alpha_4}}\mathcal{L}\{w_1\}-\frac{d}{s^{\alpha_4}}\mathcal{L}\{w_1\}\\ \vdots\\ \mathcal{L}\{u_{n+1}\}=\frac{d}{s^{\alpha_1}}\mathcal{L}\{u_n\}-\frac{\beta}{s^{\alpha_1}}\mathcal{L}\{P_n\}-\frac{\acute{\beta}}{s^{\alpha_1}}\mathcal{L}\{Q_n\},\mathcal{L}\{v_{n+1}\}=\frac{\beta}{s^{\alpha_2}}\mathcal{L}\{P_n\}\\ -\left(\frac{\gamma+d+c}{s^{\alpha_2}}\right)\mathcal{L}\{v_n\},\mathcal{L}\{\acute{v}_{n+1}\}=\frac{\acute{\beta}}{s^{\alpha_3}}\mathcal{L}\{Q_n\}-\left(\frac{\acute{\gamma}+d+\acute{c}}{s^{\alpha_3}}\right)\mathcal{L}\{\acute{v}_n\},\\ \mathcal{L}\{w_{n+1}\}=\frac{\gamma}{s^{\alpha_4}}\mathcal{L}\{v_n\}+\frac{\acute{\gamma}}{s^{\alpha_4}}\mathcal{L}\{w_n\}-\frac{d}{s^{\alpha_4}}\mathcal{L}\{w_n\}.\end{cases} \quad (12)$$

Taking inverse Laplace transform of (12), we get

$$u_0=N_1+\alpha N\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)},v_0=N_2,\acute{v}_0=N_3,w_0=N_4,$$
$$u_1=-N_1\left(d+\beta N_2+\acute{\beta}N_4\right)\frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)}-\alpha dN\frac{t^{2\alpha_3}}{\Gamma(2\alpha_1+1)},v_1=\left(\beta N_1-(\gamma+d+c)\right)N_2\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)},$$
$$\acute{v}_1=\left(\acute{\beta}N_1-(\acute{\gamma}+d+c)\right)N_3\frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)},w_1=(\gamma N_2+\acute{\gamma}N_3-dN_4)\frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)},$$

$$u_2 = dN_1\big(d + \beta N_2 + \acute{\beta} N_4\big)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + \alpha N d\frac{t^{3\alpha_1}}{\Gamma(3\alpha_1+1)}$$
$$- \beta N_1 N_2\big(\beta N_1 - (\gamma + d + c)\big)\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}$$
$$+ \beta N_2 N_1\left[\big(\alpha + \beta N_2 + \acute{\beta} N_4\big)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + \alpha N d\frac{t^{4\alpha_1}}{\Gamma(4\alpha_1+1)}\right]$$
$$- \acute{\beta} N_1 N_4\big[\acute{\beta} N_1 - (\acute{\gamma} + d + \acute{c})\big]\frac{t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)}$$
$$+ \acute{\beta} N_1\left[(d + \beta N_2 + N_4)\frac{t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + \alpha N d\frac{t^{4\alpha_1}}{\Gamma(4\alpha_1+1)}\right],$$
$$v_2 = N_2[\beta N_1 - (\gamma + d + c)]^2\frac{t^{2\alpha_2}}{\Gamma(2\alpha_2+1)} - \beta N_1 N_2\big[d + \beta N_2 + \acute{\beta} N_4\big]\frac{t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}$$
$$- \alpha\beta N_2 d N\frac{t^{2\alpha_1+\alpha_2}}{\Gamma(2\alpha_1+\alpha_2+1)},$$
$$\acute{v}_2 = N_3[\acute{\beta} N_2 - (\acute{\gamma} + d + c)]^2\frac{t^{2\alpha_3}}{\Gamma(2\alpha_3+1)}$$
$$- \acute{\beta} N_3 N_1\left[\big(d + \beta N_2 + \acute{\beta} N_3\big)\frac{t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)} + \alpha d N\frac{t^{2\alpha_1+\alpha_3}}{\Gamma(2\alpha_1+\alpha_3+1)}\right],$$
$$w_2 = (\beta N_1 - (\gamma + d + c))\left[\gamma N_1\frac{t^{2\alpha_4}}{\Gamma(2\alpha_4+1)} + \acute{\gamma} N_3\frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}\right]$$
$$- d(\gamma N_2 + \acute{\gamma} N_3 - d N_4)\frac{t^{2\alpha_4}}{\Gamma(2\alpha_4+1)}.$$

On the above fashion, we can obtain the remaining terms similarly. Finally, we get the solution in the form of infinite four series as given by

$$u(t) = \sum_{k=0}^{\infty} u_k(t), v(t) = \sum_{k=0}^{\infty} v_k(t), \acute{v}(t) = \sum_{k=0}^{\infty} \acute{v}_k(t), w(t) = \sum_{k=0}^{\infty} w_k(t). \qquad (13)$$

3. CONERGENCE ANALYSIS

The obtained solutions are in the form of four series, which rapidly converge. The convergence can easily be derived by using classical technique available as used in [31-33], for checking the convergence of infinite series (13). However, for sufficient condition of convergence of afore said four series, we give the following theorem.

Theorem 4.1 Let $\mathcal{E}$ and $\mathcal{E}$ be two Banach spaces and $\mathcal{F}: \mathcal{E} \to \mathcal{E}$ be a contractive nonlinear operator and $\boldsymbol{\mathcal{P}} = (u, v, \acute{v}, w)$ such that for all $\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}^* \in \mathcal{E}, \| \mathcal{F}(\boldsymbol{\mathcal{P}}) - \mathcal{F}(\boldsymbol{\mathcal{P}}^*) \| \leq \lambda \| \boldsymbol{\mathcal{P}} - \boldsymbol{\mathcal{P}}^* \|, 0 < \lambda < 1$. Then in view of Banach contraction theorem $\mathcal{F}$ has a unique fixed point $\boldsymbol{\mathcal{P}}$ such that $\mathcal{F}\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{P}}$, where $\boldsymbol{\mathcal{P}} = (u, v, \acute{v}, w)$.

Let us write the generated series (13), by the aforementioned Laplace Adomian decomposition method (LADM) as

$$\boldsymbol{\mathcal{P}}_k = \mathcal{F}(\boldsymbol{u}_{k-1}), \boldsymbol{\mathcal{P}}_{k-1} = \sum_{j=1}^{k-1} \boldsymbol{p}_i, k = 1,2,3, \ldots,$$

and suppose that $\boldsymbol{\mathcal{P}_0} = \boldsymbol{\mathcal{P}_0} \in C_{\mathcal{E}}(\boldsymbol{\mathcal{P}})$, where $\boldsymbol{C_{\mathcal{E}}}(\boldsymbol{\mathcal{P}}) = \{\boldsymbol{\mathcal{P}}^* \in \mathcal{E} : \| \boldsymbol{\mathcal{P}} - \boldsymbol{\mathcal{P}}^* \| < \epsilon\}$, then we have

(a) $\boldsymbol{\mathcal{P}_k} \in \boldsymbol{C_{\mathcal{E}}}(\boldsymbol{\mathcal{P}})$;

(b) $\lim_{k\to\infty} \boldsymbol{\mathcal{P}_k} = \boldsymbol{\mathcal{P}}$.

Proof: (a) In view of mathematical induction for $k = 1$, we have

$$\| \boldsymbol{\mathcal{P}_1} - \boldsymbol{\mathcal{P}} \| = \| \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{P}_0}) - \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{P}}) \| \leq \lambda \| \boldsymbol{\mathcal{P}_0} - \boldsymbol{\mathcal{P}} \|.$$

Assume that the result is true for $k - 1$, then

$$\| \boldsymbol{\mathcal{P}_{k-1}} - \boldsymbol{\mathcal{P}} \| \leq \lambda^{k-1} \| \boldsymbol{\mathcal{P}_0} - \boldsymbol{\mathcal{P}} \|.$$

We have

$$\| \boldsymbol{\mathcal{P}_k} - \boldsymbol{\mathcal{P}} \| = \| \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{P}_{k-1}}) - \boldsymbol{\mathcal{F}}(\boldsymbol{\mathcal{P}}) \| \leq \lambda \| \boldsymbol{\mathcal{P}_{k-1}} - \boldsymbol{\mathcal{P}} \| \leq \lambda^k \| \boldsymbol{\mathcal{P}_0} - \boldsymbol{\mathcal{P}} \|.$$

Hence we have,

$$\| \boldsymbol{\mathcal{P}_k} - \boldsymbol{\mathcal{P}} \| \leq \lambda^k \| \boldsymbol{\mathcal{P}_0} - \boldsymbol{\mathcal{P}} \| \leq \lambda^k \epsilon < \epsilon$$

hich implies that $\boldsymbol{\mathcal{P}_k} \in \boldsymbol{C_{\mathcal{E}}}(\boldsymbol{\mathcal{P}})$.

(b) As $\| \boldsymbol{\mathcal{P}_k} - \boldsymbol{\mathcal{P}} \| \leq \lambda^k \| \boldsymbol{\mathcal{P}_0} - \boldsymbol{\mathcal{P}} \|$ and as $\lim_{k\to\infty} \lambda^k = 0$. So, we have $\lim\limits_{k\to\infty} \| \boldsymbol{\mathcal{P}_k} - \boldsymbol{\mathcal{P}} \| \to 0$ which implies that $\lim_{k\to\infty} \boldsymbol{\mathcal{P}_k} = \boldsymbol{\mathcal{P}}$. □

4. NUMERICAL SIMULATION

Here, in this section, we find numerical solution of the considered model (4). Considered the population is in equilibrium with a wild type virus, then to find numerical solution in the form of infinite series by LADM, the following values are assigned to the parameters involved in the model (4).

Table1. Values of the parameters involved in the model (4).

Parameter	Description of the parameter	Parameter	Description of the parameter
$N_1 = 30$	Initial population of susceptible class	$\gamma = 0.02$	Removable rate
$N_2 = 10$	Initial population of first infected class	$c = 0.01$	Disease related death rate
$N_3 = 5$	Initial population of second infected class	$\acute{\beta} = 0.1$	Mutant virus rate
$N_4 = 15$	Initial population of recovered class	$\acute{\gamma} = 0.11$	Mutant virus rate
$\alpha = 0.1$	Birth rate	$\acute{c} = 0.2$	Mutant virus rate
$\beta = 0.01$	Virus infection rate	$d = 1.0$	Disease unrelated death rate

Then in view of the Tabel 1, the first three terms of system (3) are given as

$$\begin{cases} u_0 = 30 + \dfrac{4t^{\alpha_1}}{\Gamma(\alpha_1+1)}, v_0 = 10, \acute{v}_0 = 5, w_0 = 15 \\ u_1 = \dfrac{-18t^{\alpha_1}}{\Gamma(\alpha_1+1)} - \dfrac{0.8\, t^{2\alpha_1}}{\Gamma(2\alpha_1+1)}, v_1 = \dfrac{-0.2t^{\alpha_2}}{\Gamma(\alpha_2+1)}, \acute{v}_1 = \dfrac{1.45\, t^{\alpha_3}}{\Gamma(\alpha_3+1)}, w_1 = \dfrac{-2.75\, t^{\alpha_4}}{\Gamma(\alpha_4+1)} \\ u_2 = \dfrac{14.28\, t^{2\alpha_1}}{\Gamma(2\alpha_1+1)} + \dfrac{0.8\, t^{3\alpha_1}}{\Gamma(3\alpha_1+1)} - \dfrac{0.54\, t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} - \dfrac{2.52\, t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)} + \dfrac{2.88\, t^{4\alpha_1}}{\Gamma(4\alpha_1+1)} \\ v_2 = \dfrac{0.0004\, t^{2\alpha_2}}{\Gamma(2\alpha_2+1)} - \dfrac{1.8\, t^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)} - \dfrac{0.08\, t^{2\alpha_1+\alpha_2}}{\Gamma(2\alpha_1+\alpha_2+1)}, \\ \acute{v}_2 = \dfrac{0.4205\, t^{2\alpha_3}}{\Gamma(2\alpha_3+1)} - \dfrac{0.4\, t^{\alpha_1+\alpha_3}}{\Gamma(\alpha_1+\alpha_3+1)} - \dfrac{2.4\, t^{2\alpha_1+\alpha_3}}{\Gamma(2\alpha_1+\alpha_3+1)}, \\ w_2 = \dfrac{0.658\, t^{2\alpha_4}}{\Gamma(2\alpha_4+1)} - \dfrac{0.01\, t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)}. \end{cases} \tag{14}$$

Now, if we assign $\alpha_i = 1, i = 1,2,3,4$, then we get the series solutions for first few terms as:

$$\begin{cases} u(t) = 30 - 14t + 5.21t^2 + 1.333333334t^3 + 0.1200000000t^4, \\ v(t) = 10 - 0.2t + 1.299800000t^2 - 0.1333333334t^3, \\ \acute{v}(t) = 5 + 1.45t + 0.1025000000t^2 - 0.4000000001t^3, \\ w(t) = 15 - 2.75t + 0.3240000000t^2. \end{cases} \tag{15}$$

In same fashion, if we assign $\alpha_i = 0.95, i = 1,2,3,4$, we receive the series solutions as:

$$\begin{cases} u(t) = 30 - 14.28745427t^{0.95} + 5.702230568t^{1.90} + 1.6045778844t^{2.85} \\ \quad +1.614543269t^{3.80}, \\ v(t) = 10 - 0.2041064896t^{0.95} - 1.422602551t^{1.90} - 0.1604577884t^{2.85}, \\ \acute{v}(t) = 5 + 1.479772050t^{0.95} + 0.1121839987t^{1.90} - 0.4813733652t^{2.85}, \\ w(t) = 15 - 2.806464232t^{0.95} + 0.3546108837t^{1.90}. \end{cases} \tag{16}$$

Similarly, taking $\alpha_i = 0.95, i = 1,2,3,4$, we get

$$\begin{cases} u(t) = 30 - 14.80523957t^{0.85} + 6.745708214t^{1.70} + 0.2277092469t^{2.55} \\ \quad +0.2841328987t^{3.40}, \\ v(t) = 10 - 0.2115034224t^{0.85} - 1.682931197t^{1.70} - 0.2277092469t^{2.55}, \\ \acute{v}(t) = 5 + 1.533399812t^{0.85} + 0.1327130695t^{1.70} - 0.6831277406t^{2.55}, \\ w(t) = 15 - 2.908172058t^{0.85} + 0.4195027757t^{1.70}. \end{cases} \tag{17}$$

and taking $\alpha_i = 0.85, i = 1,2,3,4$, one has

$$\begin{cases} u(t) = 30 - 15.23291353t^{0.75} + 7.838473949t^{1.50} + 0.3138169319t^{2.25} \\ \quad +0.4800000001t^{3.00}, \\ v(t) = 10 - 0.2176130504t^{0.75} - 1.955556322t^{1.50} - 0.3138169319t^{2.25}, \\ \acute{v}(t) = 5 + 1.577694615t^{0.75} + 0.1542118195t^{1.50} - 0.9414807958t^{2.25}, \\ w(t) = 15 - 2.992179443t^{0.75} + 0.4874598003t^{1.50}. \end{cases} \tag{18}$$

Now, we plot the approximate solutions against different fractional orders in the following Figure.

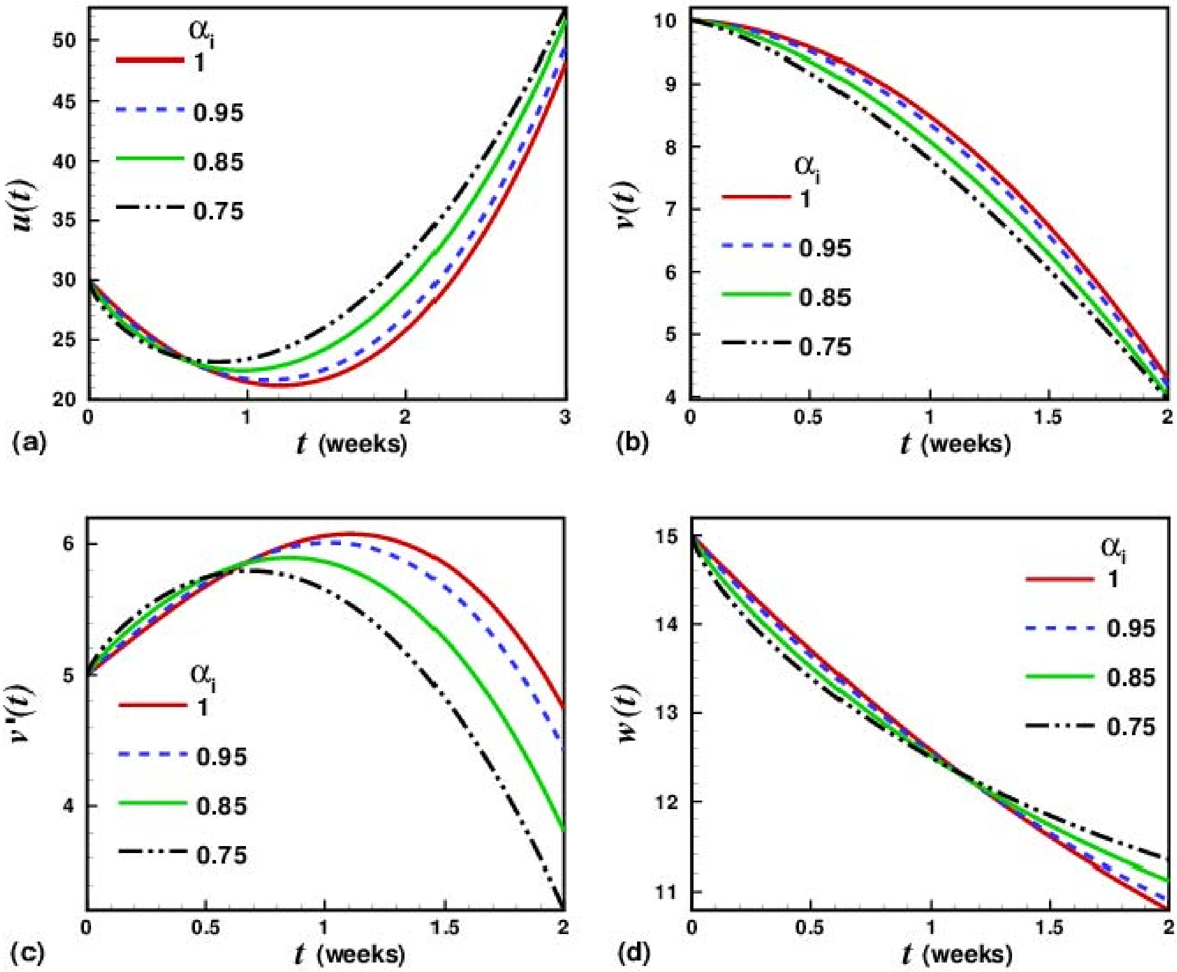

Figure 1. Plot of approximate values of various classes for different values of $\alpha_i, i = 1, 2, 3, 4$.

From the plot, we see that when the order is smaller faster the decay of susceptible $u(t)$ up to some time, then the process is inverted and the same class grows more rapidly on the same smaller order of the differentiation and vice versa, this behavior can be observed from the Figure 1 subplot (a). Similarly from the subplot (b), one can observes that smaller the fractional order fastest the decaying process of the first infected class with the passage of time and vice versa, while in Subplot (c), the second infected class initially grows at smaller order for some time but after some time at the same smaller fractional order it decays rapidly as compared to the greater fractional order. In the Subplot (b), the recovered class is decaying initially on smaller fractional order then the process become slowest after some time as compared to other fractional order and vice versa.

We compare the solution obtained by our proposed method with famous RK4 method and homotopy perturbation (HPM) method. In the following tables, we give the comparison of the proposed method with that of RK4 method. With the help of homotopy perturbation method by using classical order $\alpha_i = 1,\ i = 1,2,3,4$, we get the series solutions of the proposed model after first three terms like performed in [18], as given bellow:

$$\begin{cases} u(t) = 30 - 16t + 4.22t^2 + 0.1332t^3 + 0.13222t^4, \\ v(t) = 10 - 0.34t - 1.34213t^2 - 0.23456t^3, \\ \acute{v}(t) = 5 + 1.465t + 0.1100456t^2 - 0.421110000006t^3, \\ w(t) = 15 - 2.774t + 0.43200t^2. \end{cases} \quad (19)$$

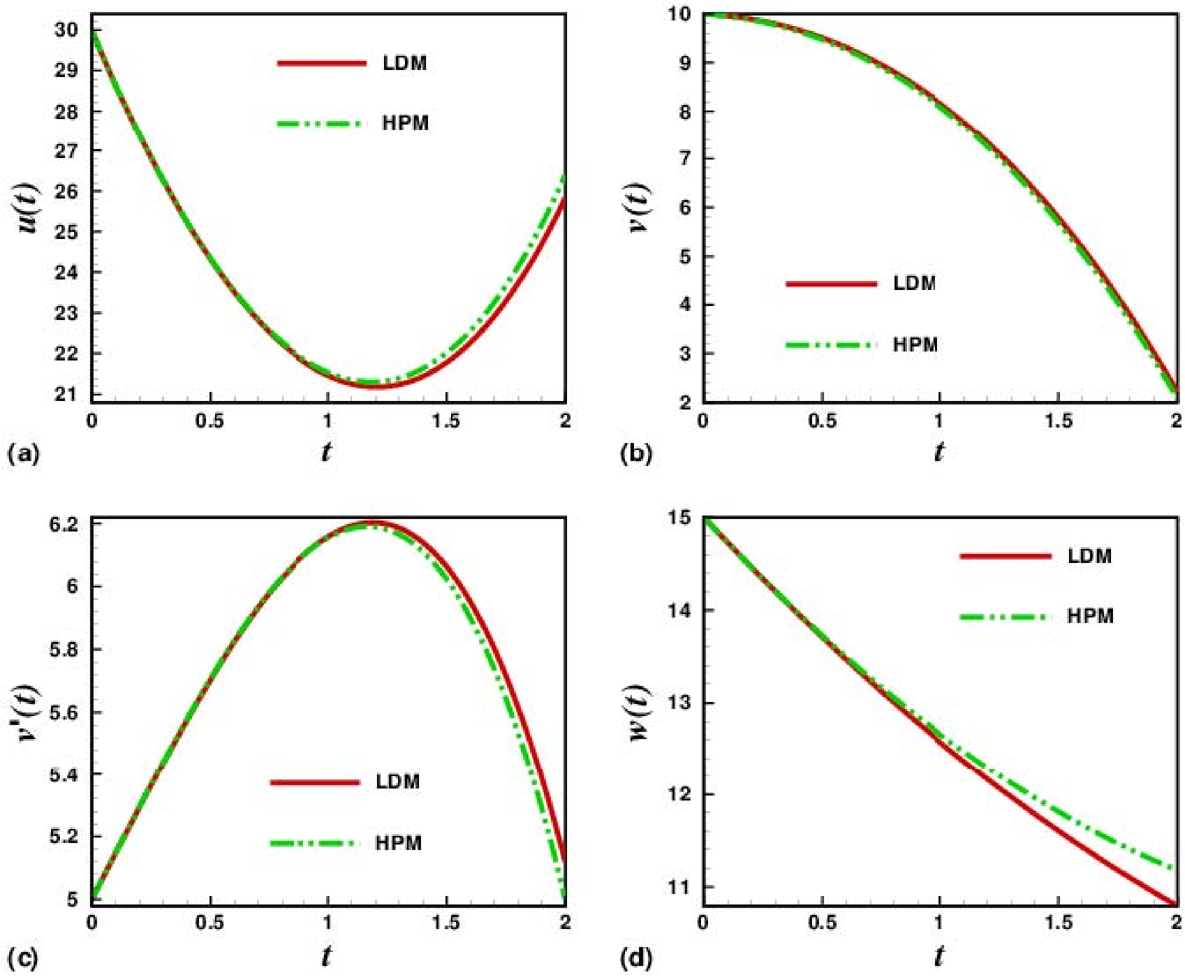

Figure 2 Comparison of solutions obtained by using proposed and homotopy perturbation method for $\alpha_i = 1, i = 1, 2, 3, 4$.

The comparison plots between different compartments are given in Fig. 2 which shows that our proposed method gives almost similar approximate solutions for the concerned model (1) to that obtained by using homotopy perturbation method up to the first three terms.

Table 2. Approximate solutions of proposed model (1) at $\alpha_i = 1, i = 1,2,3,4$.

Time (week)	$u(t)$	$v(t)$	$\acute{v}(t)$	$w(t)$
$t = 0$	30.00000	10.00000	5.00000	15.00000
$t = 0.2$	26.83943	8.57738	6.49955	12.45488
$t = 0.4$	24.33665	7.25664	6.45489	10.34709
$t = 0.6$	23.51995	6.09654	7.35295	8.64598
$t = 0.8$	22.25181	5.09710	7.27056	7.20178
$t = 1.0$	21.46333	4.19813	6.67813	6.07813
$t = 1.2$	21.14016	3.44470	5.98399	5.14242
$t = 1.4$	21.43845	3.14282	5.20121	4.37001
$t = 1.6$	22.27016	2.35215	4.45825	3.60753
$t = 1.8$	23.71771	1.92533	3.88356	3.04329
$t = 2.0$	25.82666	1.58989	3.26157	2.57085

Table 3. Approximate solutions of proposed model (1) at $\alpha_i = 1, i = 1,2,3,4$ by using RK4 method.

Time (week)	$u(t)$	$v(t)$	$\acute{v}(t)$	$w(t)$
$t = 0$	30.00000	10.00000	5.00000	15.00000
$t = 0.2$	26.93948	8.57439	6.48432	12.43030
$t = 0.4$	24.73602	7.25550	7.36831	10.34518
$t = 0.6$	23.09424	6.07969	7.59220	8.64421
$t = 0.8$	21.24964	5.05950	7.30286	7.24645
$t = 1.0$	21.37508	4.19092	6.70465	6.08937
$t = 1.2$	21.18643	3.46077	5.96843	5.12556
$t = 1.4$	21.43992	3.14234	5.20747	4.31897
$t = 1.6$	22.14428	2.34745	4.48562	3.64174
$t = 1.8$	23.54309	1.93056	3.83317	3.07190
$t = 2.0$	25.32666	1.58696	3.26024	2.59179

6. CONCLUSION

In this paper, we have considered a fractional order endemic model of non-fatal disease in a community. The concerned model was investigated for the numerical solutions via using Laplace Adomain decomposition method. The solutions obtained in the form of a series which are rapidly convergent. Also the behavior of the solutions has been verified by plotting the solutions against time for different fractional orders. Also the numerical solutions obtained by LADM with that of RK4 method in Table 2. From Table 2, one can observes that the method provides excellent numerical solutions for nonlinear fractional order models as compared to other methods like homotopy analysis and homotopy perturbation method, RK4 methods etc. Because these methods involve an extra parameter h at which the solutions depend but our propped method need no parameter and easy to understand and to implement.

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