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SOME BOUNDS FOR THE LARGEST EIGENVALUE OF WEIGHTED DISTANCE MATRIX AND WEIGHTED DISTANCE ENERGY

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Abstract. *In this study, the weighted distance matrix and the weighted distance energy for simple connected weighted graphs are considered and some bounds for the largest eigenvalue of the weighted distance matrix and the weighted distance enrgy are found. Moreover, some results are obtained by using these bounds for weighted and unweighted graphs.*

Keywords: *Weighted distance matrix, weighted distance energy, weighted Wiener index, bound*
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1. INTRODUCTION

A weighted graph is a graph that has a numeric label associated with each edge, called the weight of edge. In many applications, the edge weights are usually represented by nonnegative integers or square matrices. In this paper, we generally deal with simple connected weighted graphs where the edge weights are positive definite square matrices. Let $G=(V,E)$ be a simple connected weighted graph on *n* vertices. Let w_{ij} be the positive definite weight matrix of order *t* of the edge *ij* and assume that $w_{ij}=w_{ji}$. The weight of a vertex $i \in V$ defined as $w_i = \sum_{j:j\sim i} w_{ij}$, where *j~i* denotes the vertex *j* is adjacent to *i*.

Unless otherwise specified, by a weighted graph we mean a graph with each edge weight is a positive definite square matrix.

The weighted distance between vertices *i* and *j* of a weighted graph *G*, denoted by $D_w(i,j)$, is defined to be the sum of the weights of edges in the shortest path from *i* to *j*. Also, the weighted distance matrix $D_w(G)$ of a weighted graph *G*, is a block matrix and defined as $D_w(G)=(W_{ij})_{nt\times nt}$, where

$$W_{ij} = \begin{cases} D_w(i,j) & , \quad \text{if } i \neq j \\ 0 & , \quad \text{otherwise.} \end{cases}$$

The eigenvalues of the weihghted distance matrix are denoted by $\mu_1, \mu_2, \ldots, \mu_{nt}$. Since $D_w(G)$ is a real symmetric matrix, its eigenvalues are real and can be ordered as $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_{nt}$. Also, the largest eigenvalue of $D_w(i,j)$ is denoted by $\mu_1(D_w(i,j))$. The

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weighted transmission $Tr_w(i)$ of a vertex i in a weighted graph G is the sum of the weighted distance from i to all other vertices in G, i.e.,

$$Tr_w(i) = \sum_{\substack{j \in V \\ j \neq i}} D_w(i,j).$$

Note that the transmission of a vertex i is the sum of the entries of $D_w(G)$ in the block column (row) corresponding to i. The weihted Wiener index $W(G,w)$ of a weighted graph G is defined as

$$W(G,w) = \frac{1}{2}\sum_{i \in V}\sum_{j \in V} D_w(i,j) = \sum_{i<j} D_w(i,j).$$

The weighted distance energy of a weighted graph G is defined as the sum of the absolute values of its weighted distance eigenvalues μ_1, μ_2,..., μ_{nt} ., i.e.,

$$DE_w(G) = \sum_{i=1}^{nt} |\mu_i|.$$

The bounding problem for the largest eigenvalue of distance matrix and distance energy of unweighted graphs have received much interest. Since the fundemental paper of Ruzieh and Powers [12] in 1990, bounding problem for the largest eigenvalue of distance matrix of unweighted graphs has appeared frequently in many researches [2-4, 8, 15, 16].

The concept of distance energy for unweighted graphs introduced by Indual, Gutman and Vijayakumar [6]. The distance energy for unweighted graphs is defined as the sum of the absolute values of eigenvalues of its distance matrix. Lower and upper bounds for distance energy have been obtained in [7, 9-11, 13]. This paper is organized as follows. In Section 2, some upper bounds for the largest eigenvalue of weighted distance matrix are found. Also, some results are presented by using these bounds for number weighted and unweighted graphs. In Section 3, some upper and lower bounds for the weighted distance energy are obtained by using the definition of trace and the concept of weighted Wiener index. Moreover, some results on number weighted and unweighted graphs are found. The following lemmas are convenient for the graphs we consider. We can begin with the well-known lemma below.

Lemma 1.1 (Horn-Johnson [5]). If A is a real symmetric $n \times n$ matrix with eigenvalues $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$, then for any $\overline{x} \in \mathbb{R}^n \left(\overline{x} \neq \overline{0}\right)$, $\overline{y} \in \mathbb{R}^n \left(\overline{y} \neq \overline{0}\right)$

$$\left|\overline{x}A\overline{y}\right| \leq \mu_1 \sqrt{\overline{x}^T \overline{x}}\sqrt{\overline{y}^T \overline{y}}.$$

The equality holds if and only if $\overline{x}$ is an eigenvector of A corresponding to the largest eigenvalue μ_1 and $\overline{y} = \alpha \overline{x}$ for some $\alpha \in \mathbb{R}$.

Lemma 1.2. Let G be a simple connected weighted graph. Then

$$\mu_1(W(G,w)) \le \frac{n}{2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} tr\left(D_w(i,j)^2\right)},$$

where $\mu_1(W(G,w))$ is the largest eigenvalue of $W(G,w)$.

2. UPPER BOUNDS FOR THE LARGEST EIGENVALUE OF WEIGHTED DISTANCE MATRIX

In this section, some upper bounds for the largest eigenvalue of weighted distance matrix are presented. We can give the following theorem.

Theorem 2.1 If G be a simple connected weighted graph, then

$$\mu_1 \le \max_{i\in V}\left\{\sum_{\substack{j=1\\ j\ne i}}^{n} \mu_1(D_w(i,j))\right\}. \tag{1}$$

Proof: Let $\overline{x} = \left(\overline{x_1}^T, \overline{x_2}^T, \ldots, \overline{x_n}^T\right)^T$ be an eigenvector corresponding to μ_1 and $\overline{x_i}$ be the vector component of $\overline{x}$ such as

$$\overline{x_i}^T\,\overline{x_i} = \max_{j\in V}\left\{\overline{x_j}^T\,\overline{x_j}\right\}. \tag{2}$$

Since $\overline{x}$ is nonzero, so is $\overline{x_i}$. We have

$$D_w(G)\overline{x} = \mu_1\overline{x}. \tag{3}$$

From the i-th equation of (3), we have

$$\mu_1\overline{x_i} = \sum_{\substack{j=1\\ j\ne i}}^{n} D_w(i,j)\overline{x_j},$$

i.e.,

$$\overline{x_i}^T\mu_1\overline{x_i} \le \sum_{\substack{j=1\\ j\ne i}}^{n}\left|\overline{x_i}^T D_w(i,j)\overline{x_j}\right|.$$

From (2) and using Lemma 1.1, we get

$$\le \overline{x_i}^T\,\overline{x_i}\sum_{\substack{j=1\\ j\ne i}}^{n}\mu_1(D_w(i,j)).$$

Thus $\mu_1 \le \max_{i\in V}\left\{\sum_{\substack{j=1\\ j\ne i}}^{n}\mu_1\left(D_w(i,j)\right)\right\}$ which completes Proof.

Corollary 2.2. If *G* be a simple connected weighted graph, where each edge weight w_{ij} is a positive number, then

$$\mu_1 \le \max_{i\in V}\left\{\sum_{\substack{j=1\\ j\ne i}}^{n} D_w(i,j)\right\}.$$

Proof: For number weighted graphs, where the edge weights w_{ij} are positive number, we have $\mu_1\left(D_w(i,j)\right)=D_w(i,j)$ for all *i, j*. Using Theorem 2.1 we get the required result.

Corollary 2.3. If *G* be a simple connected unweighted graph, then

$$\mu_1 \le \max_{i\in V}\left\{\sum_{\substack{j=1\\ j\ne i}}^{n} d(i,j)\right\},$$

where $d(i,j)$ is the length of the shortest path from *i* to *j*.

Proof: For an unweighted graph, $D_w(i,j)=d(i,j)$ for all *i; j*. Using Corollary 2.2 we get the required result.

Theorem 2.4 If *G* be a simple connected weighted graph, then

$$\mu_1 \le \max_{i,j\in V}\left\{\sqrt{\sum_{\substack{k=1\\ k\ne i}}^{n}\mu_1\left(D_w(i,k)\right)\sum_{\substack{k=1\\ k\ne j}}^{n}\mu_1\left(D_w(j,k)\right)}\right\}. \tag{4}$$

Proof: Let $\overline{x}=\left(\overline{x_1}^T,\overline{x_2}^T,\ldots,\overline{x_n}^T\right)^T$ be an eigenvector corresponding to μ_1, $\overline{x_i}$ and $\overline{x_j}$ be the vector components of $\overline{x}$ such as

$$\overline{x_i}^T\,\overline{x_i}=\max_{k\in V}\left\{\overline{x_k}^T\,\overline{x_k}\right\}, \tag{5}$$

$$\overline{x_j}^T\,\overline{x_j}=\max_{k:k\ne i}\left\{\overline{x_k}^T\,\overline{x_k}\right\}. \tag{6}$$

Since $\overline{x}$ is nonzero, so is $\overline{x_i}$. We have

$$D_w(G)\overline{x}=\mu_1\overline{x}. \tag{7}$$

From the *i*-th equation of (7), we have

$$\mu_1 \overline{x_i} = \sum_{\substack{k=1 \\ k \neq i}}^{n} D_w(i,k) \overline{x_k},$$

i.e.,

$$\overline{x_i}^T \mu_1 \overline{x_i} \leq \sum_{\substack{k=1 \\ k \neq i}}^{n} \left| \overline{x_i}^T D_w(i,k) \overline{x_k} \right|.$$

From (6) and using Lemma 1.1, we get

$$\leq \sum_{\substack{k=1 \\ k \neq i}}^{n} \mu_1(D_w(i,k)) \sqrt{\overline{x_i}^T \overline{x_i}} \sqrt{\overline{x_j}^T \overline{x_j}} \ . \tag{8}$$

Similarly, from the *j*-th equation of (7), we have

$$\overline{x_j}^T \mu_1 \overline{x_j} \leq \sum_{\substack{k=1 \\ k \neq j}}^{n} \mu_1(D_w(j,k)) \sqrt{\overline{x_i}^T \overline{x_i}} \sqrt{\overline{x_j}^T \overline{x_j}} \ . \tag{9}$$

From (8) and (9), we get

$$\mu_1^2 \leq \sum_{\substack{k=1 \\ k \neq i}}^{n} \mu_1(D_w(i,k)) \sum_{\substack{k=1 \\ k \neq j}}^{n} \mu_1(D_w(j,k)).$$

Thus $\mu_1 \leq \max_{i,j \in V} \left\{ \sqrt{\sum_{\substack{k=1 \\ k \neq i}}^{n} \mu_1(D_w(i,k)) \sum_{\substack{k=1 \\ k \neq j}}^{n} \mu_1(D_w(j,k))} \right\}.$

The proof is completed.

Corollary 2.5. If *G* be a simple connected weighted graph, where each edge weight w_{ij} is a positive number, then

$$\mu_1 \leq \max_{i,j \in V} \left\{ \sqrt{\sum_{\substack{k=1 \\ k \neq i}}^{n} D_w(i,k) \sum_{\substack{k=1 \\ k \neq j}}^{n} D_w(j,k)} \right\}.$$

Proof: For number weighted graphs, where the edge weights w_{ij} are positive number, we have $\mu_1(D_w(i,j)) = D_w(i,j)$ for all *i, j*. Using Theorem 2.4 we get the required result.

Corollary 2.6. If *G* be a simple connected unweighted graph, then

$$\mu_1 \leq \max_{i,j \in V} \left\{ \sqrt{\sum_{\substack{k=1 \\ k \neq i}}^{n} d(i,k) \sum_{\substack{k=1 \\ k \neq j}}^{n} d(j,k)} \right\},$$

where $d(i,j)$ is the length of the shortest path from *i* to *j*.

Proof: For an unweighted graph, $D_w(i,j)=d(i,j)$ for all *i; j*. Using Corollary 2.5 we get the required result.

Theorem 2.7 If *G* be a simple connected weighted graph, then

$$\mu_1 \le \max_{i,j\in V}\left\{\sqrt{\sum_{\substack{k=1\\k\neq i}}^{n}\frac{\mu_1(Tr_w(k))}{\mu_1(Tr_w(i))}\mu_1(D_w(i,k))\sum_{\substack{k=1\\k\neq j}}^{n}\frac{\mu_1(Tr_w(k))}{\mu_1(Tr_w(j))}\mu_1(D_w(j,k))}\right\}. \tag{10}$$

Proof: Let us consider the matrix

$$N(G)=diag\left(\mu_1(Tr_w(1))\mathrm{I}_{t\times t},\mu_1(Tr_w(2))\mathrm{I}_{t\times t},\ldots,\mu_1(Tr_w(n))\mathrm{I}_{t\times t}\right).$$

The $(i,j)-$th element of $N(G)^{-1}D_w(G)N(G)$ is

$$\begin{cases}\dfrac{\mu_1(Tr_w(j))}{\mu_1(Tr_w(i))}D_w(i,j) & , \quad \text{if } i\neq j\\ 0 & , \quad \text{otherwise.}\end{cases}$$

Let $\overline{x}=\left(\overline{x_1}^T,\overline{x_2}^T,\ldots,\overline{x_n}^T\right)^T$ be an eigenvector corresponding to μ_1 of $N(G)^{-1}D_w(G)N(G)$, $\overline{x_i}$ and $\overline{x_j}$ be the vector components of $\overline{x}$ such as

$$\overline{x_i}^T\overline{x_i}=\max_{k\in V}\left\{\overline{x_k}^T\overline{x_k}\right\}, \tag{11}$$

$$\overline{x_j}^T\overline{x_j}=\max_{k:k\neq i}\left\{\overline{x_k}^T\overline{x_k}\right\}. \tag{12}$$

Since $\overline{x}$ is nonzero, so is $\overline{x_i}$. We have

$$\left\{N(G)^{-1}D_w(G)N(G)\right\}\overline{x}=\mu_1\overline{x}. \tag{13}$$

From the *i*-th equation of (13), we have

$$\mu_1\overline{x_i}=\sum_{\substack{k=1\\k\neq i}}^{n}\frac{\mu_1(Tr_w(k))}{\mu_1(Tr_w(i))}D_w(i,k)\overline{x_k},$$

i.e.,

$$\overline{x_i}^T\mu_1\overline{x_i}\le\sum_{\substack{k=1\\k\neq i}}^{n}\left|\overline{x_i}^T\frac{\mu_1(Tr_w(k))}{\mu_1(Tr_w(i))}D_w(i,k)\overline{x_k}\right|.$$

From (12) and using Lemma 1.1, we get

$$\le \sum_{\substack{k=1\\k\neq i}}^{n} \frac{\mu_1(Tr_w(k))}{\mu_1(Tr_w(i))}\mu_1(D_w(i,k))\sqrt{\overline{x_i}^T\overline{x_i}}\sqrt{\overline{x_j}^T\overline{x_j}}\ . \tag{14}$$

Similarly, from the j-th equation of (13), we have

$$\overline{x_j}^T\mu_1\overline{x_j} \le \sum_{\substack{k=1\\k\neq j}}^{n} \frac{\mu_1(Tr_w(k))}{\mu_1(Tr_w(j))}\mu_1(D_w(j,k))\sqrt{\overline{x_i}^T\overline{x_i}}\sqrt{\overline{x_j}^T\overline{x_j}}\ . \tag{15}$$

From (14) and (15), we get

$$\mu_1 \le \max_{i,j\in V}\left\{\sqrt{\sum_{\substack{k=1\\k\neq i}}^{n} \frac{\mu_1(Tr_w(k))}{\mu_1(Tr_w(i))}\mu_1(D_w(i,k))\sum_{\substack{k=1\\k\neq j}}^{n} \frac{\mu_1(Tr_w(k))}{\mu_1(Tr_w(j))}\mu_1(D_w(j,k))}\right\}.$$

Hence the theorem is proved.

Corollary 2.8. If G be a simple connected weighted graph, where each edge weight w_{ij} is a positive number, then

$$\mu_1 \le \max_{i,j\in V}\left\{\sqrt{\sum_{\substack{k=1\\k\neq i}}^{n} \frac{Tr_w(k)}{Tr_w(i)}D_w(i,k)\sum_{\substack{k=1\\k\neq j}}^{n} \frac{Tr_w(k)}{Tr_w(j)}D_w(j,k)}\right\}.$$

Proof: For number weighted graphs, where the edge weights w_{ij} are positive number, we have $\mu_1(D_w(i,j)) = D_w(i,j)$ and $\mu_1(Tr_w(i)) = Tr_w(i)$ for all i, j. Using Theorem 2.7 we get the required result.

Corollary 2.9. If G be a simple connected unweighted graph, then

$$\mu_1 \le \max_{i,j\in V}\left\{\sqrt{\sum_{\substack{k=1\\k\neq i}}^{n} \frac{Tr_k}{Tr_i}d(i,k)\sum_{\substack{k=1\\k\neq j}}^{n} \frac{Tr_k}{Tr_j}d(j,k)}\right\},$$

where $d(i,j)$ is the length of the shortest path from i to j and Tr_i is the sum of the distances from i to all other vertices in G.

Proof: For an unweighted graph, $D_w(i,j) = d(i,j)$ and $Tr_w(i) = Tr_i$ for all $i; j$.
Using Corollary 2.8 we get the required result.

3. BOUNDS FOR THE WEIGHTED DISTANCE ENERGY

In this section, some upper and lower bounds for the weighted distance energy are found. Firstly, we present the following lemma.

Lemma 3.1. If *G* be a simple connected weighted graph, $\mu_1 \geq \mu_2 \geq ... \geq \mu_{nt}$ be the eigenvalues of $D_w(G)$ then

$$\sum_{i=1}^{nt} \mu_i = 0$$

and

$$\sum_{i=1}^{nt} \mu_i^{\ 2} = 2\sum_{i<j} tr\left(D_w(i,j)^2\right),$$

where $tr\left(D_w(i,j)^2\right)$ is the trace of $D_w(i,j)^2$.

Proof: From the denition of trace, we get

$$\sum_{i=1}^{nt} \mu_i = tr[D_w(G)] = 0.$$

Let us consider the matrix $D_w(G)^2$.The $(i,i)-$the element of $D_w(G)^2$ is $\sum_{\substack{j=1\\ j\neq i}}^{n} D_w(i,j)^2$.

Thus,

$$\sum_{i=1}^{nt} \mu_i^{\ 2} = tr\left[D_w(G)^2\right] = 2\sum_{i<j} tr\left(D_w(i,j)^2\right).$$

This completes the Proof.

Theorem 3.2. If *G* be a simple connected weighted graph, then

$$\sqrt{2\sum_{i<j} tr\left(D_w(i,j)^2\right)} \leq DE_w(G) \leq \sqrt{2nt\sum_{i<j} tr\left(D_w(i,j)^2\right)}. \tag{16}$$

Proof: From the definition of weighted distance energy, using Lemma 3.1 and Cauchy-Schwartz inequality, we get

$$\begin{aligned}(DE_w(G))^2 &= \left(\sum_{i=1}^{nt} |\mu_i|\right)^2 \\ &\leq nt\sum_{i=1}^{nt} \mu_i^2 \\ &\leq 2nt\sum_{i<j} tr\left(D_w(i,j)^2\right).\end{aligned} \tag{17}$$

On the other hand, from the definition of weighted distance energy, we have

$$\begin{aligned}(DE_w(G))^2 = \left(\sum_{i=1}^{nt} |\mu_i|\right)^2 &\geq \sum_{i=1}^{nt} \mu_i^2 \\ &= 2\sum_{i<j} tr\left(D_w(i,j)^2\right).\end{aligned} \tag{18}$$

From (17) and (18), we get

$$\sqrt{2\sum_{i<j} tr\left(D_w(i,j)^2\right)} \leq DE_w(G) \leq \sqrt{2nt\sum_{i<j} tr\left(D_w(i,j)^2\right)},$$

so the proof is completed.

Corollary 3.3. If *G* be a simple connected weighted graph, where each edge weight w_{ij} is a positive number, then

$$\sqrt{2\sum_{i<j} D_w(i,j)^2} \le DE_w(G) \le \sqrt{2n\sum_{i<j} D_w(i,j)^2} .$$

Proof: For number weighted graphs, where the edge weights w_{ij} are positive number, we have $tr\left(D_w(i,j)^2\right) = D_w(i,j)^2$ for all *i, j*. Using Theorem 3.2 we get the required result.

Corollary 3.4. If *G* be a simple connected unweighted graph, then

$$\sqrt{2\sum_{i<j} d(i,j)^2} \le DE_w(G) \le \sqrt{2n\sum_{i<j} d(i,j)^2} ,$$

where $d(i,j)$ is the length of the shortest path from *i* to *j*.

Proof: For an unweighted graph, $D_w(i,j) = d(i,j)$ for all *i; j*. Using Corollary 3.3 we get the required result.

Theorem 3.5. If *G* be a simple connected weighted graph, then

$$DE_w(G) \le \mu_1 + \sqrt{(nt-1)\left(2\sum_{i<j} tr\left(D_w(i,j)^2\right) - \mu_1^{\,2}\right)} . \tag{19}$$

Proof: By the definition of weighted distance energy and using Cauchy-Schwartz inequality and Lemma 3.1, we get

$$\left(DE_w(G) - \mu_1\right)^2 = \left(\sum_{i=2}^{nt} |\mu_i|\right)^2 \le (nt-1)\left(\sum_{i=1}^{nt} \mu_i^2 - \mu_1^2\right)$$

$$\le (nt-1)\left(2\sum_{i<j} tr\left(D_w(i,j)^2\right) - \mu_1^2\right)$$

and then

$$DE_w(G) \le \mu_1 + \sqrt{(nt-1)\left(2\sum_{i<j} tr\left(D_w(i,j)^2\right) - \mu_1^{\,2}\right)} .$$

Hence the theorem is proved.

Corollary 3.6. If *G* be a simple connected weighted graph, where each edge weight w_{ij} is a positive number, then

$$DE_w(G) \le \mu_1 + \sqrt{(n-1)\left(2\sum_{i<j} D_w(i,j)^2 - \mu_1^{\,2}\right)}.$$

Proof: For number weighted graphs, where the edge weights w_{ij} are positive number, we have $tr\left(D_w(i,j)^2\right) = D_w(i,j)^2$ for all *i, j*. Using Theorem 3.5 we get the required result.

Corollary 3.7. If *G* be a simple connected unweighted graph, then

$$DE_w(G) \le \mu_1 + \sqrt{(n-1)\left(2\sum_{i<j} d(i,j)^2 - \mu_1^{\,2}\right)},$$

where $d(i,j)$ is the length of the shortest path from *i* to *j*.

Proof: For an unweighted graph, $D_w(i,j) = d(i,j)$ for all *i; j*. Using Corollary 3.6 we get the required result.

Theorem 3.8. If *G* be a simple connected weighted graph, then

$$\frac{2}{n}\mu_1(W(G,w)) \le DE_w(G), \tag{20}$$

where $\mu_1(W(G,w))$ is the largest eigenvalue of $W(G,w)$.

Proof: Using Theorem 3.2 and Lemma 1.2, we get

$$\begin{aligned} DE_w(G)^2 &\ge 2\sum_{i<j} tr\left(D_w(i,j)^2\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} tr\left(D_w(i,j)^2\right) \\ &= \frac{4}{n^2}\left(\frac{n^2}{4}\sum_{i=1}^{n}\sum_{j=1}^{n} tr\left(D_w(i,j)^2\right)\right) \\ &= \frac{4}{n^2}\mu_1^2\left(W(G,w)\right) \end{aligned}$$

and then

$$DE_w(G) \ge \frac{2}{n}\mu_1(W(G,w)).$$

The proof is completed.

Corollary 3.9. If *G* be a simple connected weighted graph, where each edge weight w_{ij} is a positive number, then

$$DE_w(G) \ge \frac{2}{n}W(G,w).$$

Proof: For number weighted graphs, where the edge weights w_{ij} are positive number, we have $\mu_1(W(G,w)) = W(G,w)$. Using Theorem 3.8 we get the required result.

Corollary 3.10. If G be a simple connected unweighted graph, then

$$DE_w(G) \geq \frac{2}{n} W ,$$

where W is the Wiener index for unweighted graphs.
Proof: For an unweighted graph, $W(G,w) = W$ .Using Corollary 3.9 we get the required result.

Theorem 3.11. If G be a simple connected weighted graph, then

$$DE_w(G) \geq \sqrt{2\sum_{i<j} tr\left(D_w(i,j)^2\right) + nt(nt-1)\det\left(D_w(G)\right)^{\frac{2}{nt}}} . \quad (21)$$

Proof: From the definition of weighted distance energy and using Lemma 3.1, we get

$$DE_w(G)^2 = \left(\sum_{i=1}^{nt} |\mu_i|\right)^2 = \sum_{i=1}^{nt} \mu_i^2 + 2\sum_{i<j} |\mu_i||\mu_j|$$
$$= 2\sum_{i<j} tr\left(D_w(i,j)^2\right) + \sum_{i\neq j} |\mu_i||\mu_j| .$$

Since for nonnegative numbers the geometric mean smaller than the arithmetic mean. Thus, we have

$$= 2\sum_{i<j} tr\left(D_w(i,j)^2\right) + nt(nt-1)\left(\frac{1}{nt(nt-1)}\sum_{i\neq j} |\mu_i||\mu_j|\right)$$
$$\geq 2\sum_{i<j} tr\left(D_w(i,j)^2\right) + nt(nt-1)\left(\prod_{i\neq j} |\mu_i||\mu_j|\right)^{\frac{1}{nt(nt-1)}}$$
$$= 2\sum_{i<j} tr\left(D_w(i,j)^2\right) + nt(nt-1)\left(\prod_{i=1}^{nt} |\mu_i|^{2(nt-1)}\right)^{\frac{1}{nt(nt-1)}} .$$
$$= 2\sum_{i<j} tr\left(D_w(i,j)^2\right) + nt(nt-1)\prod_{i=1}^{nt} |\mu_i|^{\frac{2}{nt}}$$
$$= 2\sum_{i<j} tr\left(D_w(i,j)^2\right) + nt(nt-1)\det\left(D_w(G)\right)^{\frac{2}{nt}}$$

and then

$$DE_w(G) \geq \sqrt{2\sum_{i<j} tr\left(D_w(i,j)^2\right) + nt(nt-1)\det\left(D_w(G)\right)^{\frac{2}{nt}}} .$$

Hence the theorem is proved.

Corollary 3.12. If G be a simple connected weighted graph, where each edge weight w_{ij} is a positive number, then

$$DE_w(G) \geq \sqrt{2\sum_{i<j} D_w(i,j)^2 + n(n-1)\det(D_w(G))^{\frac{2}{n}}}\ .$$

Proof: For number weighted graphs, where the edge weights w_{ij} are positive number, we have $tr\left(D_w(i,j)^2\right) = D_w(i,j)^2$ for all *i, j*. Using Theorem 3.11 we get the required result.

Corollary 3.13. If *G* be a simple connected unweighted graph, then

$$DE_w(G) \geq \sqrt{2\sum_{i<j} d(i,j)^2 + n(n-1)\det(D(G))^{\frac{2}{n}}}\ ,$$

where $d(i,j)$ is the length of the shortest path from *i* to *j* and $D(G)$ is the distance matrix of *G*.

Proof: For an unweighted graph, $D_w(G) = D(G)$ and $D_w(i,j) = d(i,j)$ for all *i; j*. Using Corollary 3.12 we get the required result.

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