ORIGINAL PAPER

ON NANO SEMI ALPHA OPEN SETS

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Abstract. *In this paper, we presented another concept of N-$O.S.$ called NS_{α}-$O.S.$ and studied their fundamental properties in nano topological spaces. We also present NS_{α}-interior and NS_{α}-closure and study some of their fundamental properties.*

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1. INTRODUCTION

In 2000, G.B. Navalagi [1] presented the idea of semi-α-open sets in topological spaces. N.M. Ali [2] introduced new types of weakly open sets in topological spaces. M.L. Thivagar and C. Richard [3] gave nano topological space (or simply $\mathrm{N.T.S.}$) on a subset $\mathcal{M}$ of a universe which is defined regarding lower and upper approximations of $\mathcal{M}$. He studied about the weak forms of nano open sets (briefly N-$\mathrm{O.S.}$), such as $N\alpha$-$\mathrm{O.S.}$, Ns-$\mathrm{O.S.}$, and Np-$\mathrm{O.S.}$. The objective of this paper is to present the idea of NS_{α}-$\mathrm{O.S.}$ and study their fundamental properties in nano topological spaces. We also present NS_{α}-interior and NS_{α}-closure and obtain some of its properties.

2. PRELIMINARIES

Throughout this paper, $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ (or simply $\mathcal{U}$) always mean a nano topological space on which no separation axioms are expected unless generally specified. The complement of a N-$\mathrm{O.S.}$ is called a nano closed set (briefly N-$\mathrm{C.S.}$) in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$. For a set $\mathcal{C}$ in a nano topological space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, $Ncl(\mathcal{C})$, $Nint(\mathcal{C})$ and $\mathcal{C}^c = \mathcal{U} - \mathcal{C}$ denote the nano closure of $\mathcal{C}$, the nano interior of $\mathcal{C}$ and the nano complement of $\mathcal{C}$ respectively.

Definition 2.1 [3]:

A subset $\mathcal{C}$ of an $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is said to be:
(i) A nano pre-open set (briefly Np-$\mathrm{O.S.}$) if $\mathcal{C} \subseteq Nint(Ncl(\mathcal{C}))$. The complement of a Np-$\mathrm{O.S.}$ is called a nano pre-closed set (briefly Np-$\mathrm{C.S.}$) in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$. The family of all Np-$\mathrm{O.S.}$ (resp. Np-$\mathrm{C.S.}$) of $\mathcal{U}$ is denoted by $NpO(\mathcal{U}, \mathcal{M})$ (resp. $NpC(\mathcal{U}, \mathcal{M})$).
(ii) A nano semi-open set (briefly Ns-$\mathrm{O.S.}$) if $\mathcal{C} \subseteq Ncl(Nint(\mathcal{C}))$. The complement of a Ns-$\mathrm{O.S.}$ is called a nano semi-closed set (briefly Ns-$\mathrm{C.S.}$) in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$. The family of all Ns-$\mathrm{O.S.}$ (resp. Ns-$\mathrm{C.S.}$) of $\mathcal{U}$ is denoted by $NsO(\mathcal{U}, \mathcal{M})$ (resp. $NsC(\mathcal{U}, \mathcal{M})$).
(iii) A nano α-open set (briefly $N\alpha$-$\mathrm{O.S.}$) if $\mathcal{C} \subseteq Nint(Ncl(Nint(\mathcal{C})))$. The complement of a $N\alpha$-$\mathrm{O.S.}$ is called a nano α-closed set (briefly $N\alpha$-$\mathrm{C.S.}$) in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$. The family of all $N\alpha$-$\mathrm{O.S.}$ (resp. $N\alpha$-$\mathrm{C.S.}$) of $\mathcal{U}$ is denoted by $N\alpha O(\mathcal{U}, \mathcal{M})$ (resp. $N\alpha C(\mathcal{U}, \mathcal{M})$).

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Definition 2.2 [3]:

(i) The Np-interior of a set $\mathcal{C}$ of a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is the union of all Np-$\mathrm{O.S.}$ contained in $\mathcal{C}$ and is denoted by $Npint(\mathcal{C})$.
(ii) The Ns-interior of a set $\mathcal{C}$ of a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is the union of all Ns-$\mathrm{O.S.}$ contained in $\mathcal{C}$ and is denoted by $Nsint(\mathcal{C})$.
(iii) The $N\alpha$-interior of a set $\mathcal{C}$ of a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is the union of all $N\alpha$-$\mathrm{O.S.}$ contained in $\mathcal{C}$ and is denoted by $N\alpha int(\mathcal{C})$.

Definition 2.3 [3]:

(i) The Np-closure of a set $\mathcal{C}$ of a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is the intersection of all Np-$\mathrm{C.S.}$ that contain $\mathcal{C}$ and is denoted by $Npcl(\mathcal{C})$.
(ii) The Ns-closure of a set $\mathcal{C}$ of a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is the intersection of all Ns-$\mathrm{C.S.}$ that contain $\mathcal{C}$ and is denoted by $Nscl(\mathcal{C})$.
(iii) The $N\alpha$-closure of a set $\mathcal{C}$ of a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is the intersection of all $N\alpha$-$\mathrm{C.S.}$ that contain $\mathcal{C}$ and is denoted by $N\alpha cl(\mathcal{C})$.

Proposition 2.4 [3]:

In a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, then the following statements hold, and the equality of each statement are not true:
(i) Every N-$\mathrm{O.S.}$ (resp. N-$\mathrm{C.S.}$) is a $N\alpha$-$\mathrm{O.S.}$ (resp. $N\alpha$-$\mathrm{C.S.}$).
(ii) Every $N\alpha$-$\mathrm{O.S.}$ (resp. $N\alpha$-$\mathrm{C.S.}$) is a Ns-$\mathrm{O.S.}$ (resp. Ns-$\mathrm{C.S.}$).
(iii) Every $N\alpha$-$\mathrm{O.S.}$ (resp. $N\alpha$-$\mathrm{C.S.}$) is a Np-$\mathrm{O.S.}$ (resp. Np-$\mathrm{C.S.}$).

Proposition 2.5 [3]:

A subset $\mathcal{C}$ of a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is a $N\alpha$-$\mathrm{O.S.}$ iff $\mathcal{C}$ is a Ns-$\mathrm{O.S.}$ and Np-$\mathrm{O.S.}$.

Lemma 2.6:

(i) If $\mathcal{K}$ is a N-$\mathrm{O.S.}$, then $Nscl(\mathcal{K}) = Nint(Ncl(\mathcal{K}))$.
(ii) If $\mathcal{C}$ is a subset of a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, then $Nsint\big(Ncl(\mathcal{C})\big) = Ncl(Nint(Ncl(\mathcal{C})))$.

3. NANO SEMI-α-OPEN SETS

In this section, we present and study the NS_{α}-$\mathrm{O.S.}$ and some of its properties.

Definition 3.1:

A subset $\mathcal{C}$ of a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is called nano semi-α-open set (briefly NS_{α}-$\mathrm{O.S.}$) if there exists a $N\alpha$-$\mathrm{O.S.}$ $\mathcal{P}$ in $\mathcal{U}$ such that $\mathcal{P} \subseteq \mathcal{C} \subseteq Ncl(\mathcal{P})$ or equivalently if $\mathcal{C} \subseteq Ncl(N\alpha int(\mathcal{C}))$. The family of all NS_{α}-$\mathrm{O.S.}$ of $\mathcal{U}$ is denoted by $NS_{\alpha}O(\mathcal{U}\ , \mathcal{M})$.

Definition 3.2:

The complement of NS_{α}-$\mathrm{O.S.}$ is called a nano semi-α-closed set (briefly NS_{α}-$\mathrm{C.S.}$). The family of all NS_{α}-$\mathrm{C.S.}$ of $\mathcal{U}$ is denoted by $NS_{\alpha}C(\mathcal{U}\ , \mathcal{M})$.

Example 3.3:

Let $\mathcal{U} = \{p, q, r, s\}$ with $\mathcal{U}/\mathcal{R} = \{\{p\}, \{r\}, \{q, s\}\}$ and $\mathcal{M} = \{p, q\}$.
Let $\tau_{\mathcal{R}}(\mathcal{M}) = \{\phi, \{p\}, \{q, s\}, \{p, q, s\}, \mathcal{U}\}$ be a $\mathrm{N.T.S.}$. The N-$\mathrm{C.S.}$ are $\mathcal{U}, \{q, r, s\}, \{p, r\}, \{r\}$ and ϕ. The family of all $N\alpha$-$\mathrm{O.S.}$ of $\mathcal{U}$ is: $N\alpha O(\mathcal{U}, \mathcal{M}) = \{\phi, \{p\}, \{q, s\}, \{p, q, s\}, \mathcal{U}\}$.
The family of all $N\alpha$-$\mathrm{C.S.}$ of $\mathcal{U}$ is: $N\alpha C(\mathcal{U}, \mathcal{M}) = \{\mathcal{U}, \{q, r, s\}, \{p, r\}, \{r\}, \phi\}$.
The family of all NS_α-$\mathrm{O.S.}$ of $\mathcal{U}$ is: $NS_\alpha O(\mathcal{U}, \mathcal{M}) = N\alpha O(\mathcal{U}, \mathcal{M}) \cup \{\{p, r\}, \{q, r, s\}\}$.
The family of all NS_α-$\mathrm{C.S.}$ of $\mathcal{U}$ is: $NS_\alpha C(\mathcal{U}, \mathcal{M}) = N\alpha C(\mathcal{U}, \mathcal{M}) \cup \{\{q, s\}, \{p\}\}$.

Remark 3.4:

It is evident by definitions that in a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, the following hold:
(i) Every N-$\mathrm{O.S.}$ (resp. N-$\mathrm{C.S.}$) is a NS_α-$\mathrm{O.S.}$ (resp. NS_α-$\mathrm{C.S.}$).
(ii) Every $N\alpha$-$\mathrm{O.S.}$ (resp. $N\alpha$-$\mathrm{C.S.}$) is a NS_α-$\mathrm{O.S.}$ (resp. NS_α-$\mathrm{C.S.}$).

The opposite of the above remark need not be true as appeared in the following examples.

Example 3.5:

In example (3.3), the set $\{p, r\}$ is a NS_α-$\mathrm{O.S.}$ but is not N-$\mathrm{O.S.}$ and not $N\alpha$-$\mathrm{O.S.}$. The set $\{q, s\}$ is a NS_α-$\mathrm{C.S.}$ but is not N-$\mathrm{C.S.}$ and not $N\alpha$-$\mathrm{C.S.}$.

Remark 3.6:

The concepts of NS_α-$\mathrm{O.S.}$ and Np-$\mathrm{O.S.}$ are independent, as the following example shows.

Example 3.7:

In example (3.3), then the set $\{p, r\}$ is a NS_α-$\mathrm{O.S.}$ but is not Np-$\mathrm{O.S.}$. The set $\{p, r, s\}$ is a Np-$\mathrm{O.S.}$ but is not NS_α-$\mathrm{O.S.}$.

Remark 3.8:

(i) If every N-$\mathrm{O.S.}$ is a N-$\mathrm{C.S.}$ and every nowhere nano dense set is N-$\mathrm{C.S.}$ in any $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, then every NS_α-$\mathrm{O.S.}$ is a N-$\mathrm{O.S.}$.
(ii) If every N-$\mathrm{O.S.}$ is a N-$\mathrm{C.S.}$ in any $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, then every NS_α-$\mathrm{O.S.}$ is a $N\alpha$-$\mathrm{O.S.}$.

Remark 3.9:

(i) It is clear that every Ns-$\mathrm{O.S.}$ and Np-$\mathrm{O.S.}$ of any $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is a NS_α-$\mathrm{O.S.}$ (by proposition (2.5) and remark (3.4) (ii)).
(ii) A NS_α-$\mathrm{O.S.}$ in any $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ is a Np-$\mathrm{O.S.}$ if every N-$\mathrm{O.S.}$ of $\mathcal{U}$ is a N-$\mathrm{C.S.}$ (from proposition (2.4) (iii) and remark (3.8) (ii)).

Theorem 3.10:

For any subset $\mathcal{C}$ of a $\mathrm{N.T.S.}$ $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, $\mathcal{C} \in N\alpha O(\mathcal{U}, \mathcal{M})$ iff there exists a N-$\mathrm{O.S.}$ $\mathcal{K}$ such that $\mathcal{K} \subseteq \mathcal{C} \subseteq Nint(Ncl(\mathcal{K}))$.

Proof: Let $\mathcal{C}$ be a $N\alpha$-$\mathrm{O.S.}$. Hence $\mathcal{C} \subseteq Nint(Ncl(Nint(\mathcal{C})))$, so let $\mathcal{K} = Nint(\mathcal{C})$, we get $Nint(\mathcal{C}) \subseteq \mathcal{C} \subseteq Nint(Ncl(Nint(\mathcal{C})))$. Then there exists a N-$\mathrm{O.S.}$ $Nint(\mathcal{C})$ such that $\mathcal{K} \subseteq \mathcal{C} \subseteq Nint(Ncl(\mathcal{K}))$, where $\mathcal{K} = Nint(\mathcal{C})$.

Conversely, suppose that there is a N-O.S. $\mathcal{K}$ such that $\mathcal{K} \subseteq \mathcal{C} \subseteq Nint(Ncl(\mathcal{K}))$.
To prove $\mathcal{C} \in N\alpha O(\mathcal{U}, \mathcal{M})$.
$\mathcal{K} \subseteq Nint(\mathcal{C})$ (since $Nint(\mathcal{C})$ is the largest N-O.S. contained in $\mathcal{C}$).
Hence $Ncl(\mathcal{K}) \subseteq Nint(Ncl(\mathcal{C}))$, then $Nint(Ncl(\mathcal{K})) \subseteq Nint(Ncl(Nint(\mathcal{C})))$.
But $\mathcal{K} \subseteq \mathcal{C} \subseteq Nint(Ncl(\mathcal{K}))$ (by hypothesis). Then $\mathcal{C} \subseteq Nint(Ncl(Nint(\mathcal{C})))$.
Therefore, $\mathcal{C} \in N\alpha O(\mathcal{U}, \mathcal{M})$.

Theorem 3.11:

For any subset $\mathcal{C}$ of a N.T.S. $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$. The following properties are equivalent:
(i) $\mathcal{C} \in NS_\alpha O(\mathcal{U}, \mathcal{M})$.
(ii) There exists a N-O.S. say $\mathcal{K}$ such that $\mathcal{K} \subseteq \mathcal{C} \subseteq Ncl(Nint(Ncl(\mathcal{K})))$.
(iii) $\mathcal{C} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$.

Proof:
$(i) \Longrightarrow (ii)$ Let $\mathcal{C} \in NS_\alpha O(\mathcal{U}, \mathcal{M})$. Then there exists $\mathcal{P} \in N\alpha O(\mathcal{U}, \mathcal{M})$, such that $\mathcal{P} \subseteq \mathcal{C} \subseteq Ncl(\mathcal{P})$. Hence there exists $\mathcal{K}$ N-O.S. such that $\mathcal{K} \subseteq \mathcal{P} \subseteq Nint(Ncl(\mathcal{K}))$ (by theorem (3.10)). Therefore, $Ncl(\mathcal{K}) \subseteq Ncl(\mathcal{P}) \subseteq Ncl(Nint(Ncl(\mathcal{K})))$, implies that $Ncl(\mathcal{P}) \subseteq Ncl(Nint(Ncl(\mathcal{K})))$. Then $\mathcal{K} \subseteq \mathcal{P} \subseteq \mathcal{C} \subseteq Ncl(\mathcal{P}) \subseteq Ncl(Nint(Ncl(\mathcal{K})))$. Therefore, $\mathcal{K} \subseteq \mathcal{C} \subseteq Ncl(Nint(Ncl(\mathcal{K})))$, for some $\mathcal{K}$ N-O.S..
$(ii) \Longrightarrow (iii)$ Suppose that there exists a N-O.S. $\mathcal{K}$ such that $\mathcal{K} \subseteq \mathcal{C} \subseteq Ncl(Nint(Ncl(\mathcal{K})))$. We know that $Nint(\mathcal{C}) \subseteq \mathcal{C}$. On the other hand, $\mathcal{K} \subseteq Nint(\mathcal{C})$ (since $Nint(\mathcal{C})$ is the largest N-O.S. contained in $\mathcal{C}$). Hence $Ncl(\mathcal{K}) \subseteq Ncl(Nint(\mathcal{C}))$, then $Nint(Ncl(\mathcal{K})) \subseteq Nint(Ncl(Nint(\mathcal{C})))$, therefore $Ncl(Nint(Ncl(\mathcal{K}))) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$.
But $\mathcal{C} \subseteq Ncl(Nint(Ncl(\mathcal{K})))$ (by hypothesis).
Hence $\mathcal{C} \subseteq Ncl(Nint(Ncl(\mathcal{K}))) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$,
then $\mathcal{C} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$.
$(iii) \Longrightarrow (i)$ Let $\mathcal{C} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$. To prove $\mathcal{C} \in NS_\alpha O(\mathcal{U}, \mathcal{M})$.
Let $\mathcal{P} = Nint(\mathcal{C})$; we know that $Nint(\mathcal{C}) \subseteq \mathcal{C}$. To prove $\mathcal{C} \subseteq Ncl(Nint(\mathcal{C}))$.
Since $Nint(Ncl(Nint(\mathcal{C}))) \subseteq Ncl(Nint(\mathcal{C}))$.
Hence, $Ncl(Nint(Ncl(Nint(\mathcal{C})))) \subseteq Ncl(Ncl(Nint(\mathcal{C})))) = Ncl(Nint(\mathcal{C}))$.
But $\mathcal{C} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$ (by hypothesis).
Hence, $\mathcal{C} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C})))) \subseteq Ncl(Nint(\mathcal{C})) \Longrightarrow \mathcal{C} \subseteq Ncl(Nint(\mathcal{C}))$.
Hence, there exists a N-O.S. say $\mathcal{P}$, such that $\mathcal{P} \subseteq \mathcal{C} \subseteq Ncl(\mathcal{P})$.
On the other hand, $\mathcal{P}$ is a $N\alpha$-O.S. (since $\mathcal{P}$ is a N-O.S.). Hence $\mathcal{C} \in NS_\alpha O(\mathcal{U}, \mathcal{M})$.

Corollary 3.12:

For any subset $\mathcal{C}$ of a N.T.S. $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, the following properties are equivalent:
(i) $\mathcal{C} \in NS_\alpha C(\mathcal{U}, \mathcal{M})$.
(ii) There exists a N-C.S. $\mathcal{F}$ such that $Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{C} \subseteq \mathcal{F}$.
(iii) $Nint(Ncl(Nint(Ncl(\mathcal{C})))) \subseteq \mathcal{C}$.

Proof:
$(i) \Longrightarrow (ii)$ Let $\mathcal{C} \in NS_\alpha C(\mathcal{U}, \mathcal{M})$, then $\mathcal{C}^c \in NS_\alpha O(\mathcal{U}, \mathcal{M})$. Hence there is $\mathcal{K}$ N-O.S. such that $\mathcal{K} \subseteq \mathcal{C}^c \subseteq Ncl(Nint(Ncl(\mathcal{K})))$ (by theorem (3.11)). Hence $(Ncl(Nint(Ncl(\mathcal{K}))))^c \subseteq \mathcal{C}^{c^c} \subseteq \mathcal{K}^c$, i.e., $Nint(Ncl(Nint(\mathcal{K}^c))) \subseteq \mathcal{C} \subseteq \mathcal{K}^c$. Let $\mathcal{K}^c = \mathcal{F}$, where $\mathcal{F}$ is a N-C.S. in $\mathcal{U}$. Then $Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{C} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-C.S..
$(ii) \Longrightarrow (iii)$ Suppose that there exists $\mathcal{F}$ N-C.S. such that $Nint\left(Ncl\big(Nint(\mathcal{F})\big)\right) \subseteq \mathcal{C} \subseteq \mathcal{F}$, but $Ncl(\mathcal{C})$ is the smallest N-C.S. containing $\mathcal{C}$. Then $Ncl(\mathcal{C}) \subseteq \mathcal{F}$, and therefore:

$Nint\big(Ncl(\mathcal{C})\big) \subseteq Nint(\mathcal{F}) \Longrightarrow Ncl\Big(Nint\big(Ncl(\mathcal{C})\big)\Big) \subseteq Ncl\big(Nint(\mathcal{F})\big) \Longrightarrow$
$Nint(Ncl(Nint(Ncl(\mathcal{C})))) \subseteq Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{C} \Longrightarrow$
$Nint(Ncl(Nint(Ncl(\mathcal{C})))) \subseteq \mathcal{C}$.
$(iii) \Longrightarrow (i)$ Let $Nint(Ncl(Nint(Ncl(\mathcal{C})))) \subseteq \mathcal{C}$. To prove $\mathcal{C} \in NS_{\alpha}C(\mathcal{U}, \mathcal{M})$,
i.e., to prove $\mathcal{C}^c \in NS_{\alpha}O(\mathcal{U}, \mathcal{M})$.
Then $\mathcal{C}^c \subseteq (Nint(Ncl(Nint(Ncl(\mathcal{C})))))^c = Ncl(Nint(Ncl(Nint(\mathcal{C}^c))))$, but
$(Nint(Ncl(Nint(Ncl(\mathcal{C})))))^c = Ncl(Nint(Ncl(Nint(\mathcal{C}^c))))$.
Hence $\mathcal{C}^c \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}^c))))$, and therefore $\mathcal{C}^c \in NS_{\alpha}O(\mathcal{U}, \mathcal{M})$,
i.e., $\mathcal{C} \in NS_{\alpha}C(\mathcal{U}, \mathcal{M})$.

Proposition 3.13:

The union of any family of $N\alpha$-O. S. is a $N\alpha$-O. S..

Proof: Let $\{\mathcal{C}_i\}_{i\in\Lambda}$ be a family of $N\alpha$-O. S. of $\mathcal{U}$. To prove $\bigcup_{i\in\Lambda}\mathcal{C}_i$ is a $N\alpha$-O. S.,
i.e., $\bigcup_{i\in\Lambda}\mathcal{C}_i \subseteq Nint(Ncl(Nint(\bigcup_{i\in\Lambda}\mathcal{C}_i)))$. Then $\mathcal{C}_i \subseteq Nint(Ncl(Nint(\mathcal{C}_i)))$, $\forall i \in \Lambda$.
Since $\bigcup_{i\in\Lambda} Nint(\mathcal{C}_i) \subseteq Nint(\bigcup_{i\in\Lambda}\mathcal{C}_i)$ and $\bigcup_{i\in\Lambda} Ncl(\mathcal{C}_i) \subseteq Ncl(\bigcup_{i\in\Lambda}\mathcal{C}_i)$ hold for any nano topology. We have $\bigcup_{i\in\Lambda}\mathcal{C}_i \subseteq \bigcup_{i\in\Lambda} Nint(Ncl(Nint(\mathcal{C}_i)))$

$$\subseteq Nint(\textstyle\bigcup_{i\in\Lambda} Ncl(Nint(\mathcal{C}_i)))$$
$$\subseteq Nint(Ncl(\textstyle\bigcup_{i\in\Lambda}(Nint(\mathcal{C}_i)))$$
$$\subseteq Nint(Ncl(Nint(\textstyle\bigcup_{i\in\Lambda}\mathcal{C}_i))).$$

Hence $\bigcup_{i\in\Lambda}\mathcal{C}_i$ is a $N\alpha$-O. S..

Theorem 3.14:

The union of any family of NS_{α}-O. S. is a NS_{α}-O. S..

Proof: Let $\{\mathcal{C}_i\}_{i\in\Lambda}$ be a family of NS_{α}-O. S.. To prove $\bigcup_{i\in\Lambda}\mathcal{C}_i$ is a NS_{α}-O. S..
Since $\mathcal{C}_i \in NS_{\alpha}O(\mathcal{U}, \mathcal{M})$. Then there is a $N\alpha$-O. S. $\mathcal{D}_i$ such that $\mathcal{D}_i \subseteq \mathcal{C}_i \subseteq Ncl(\mathcal{D}_i)$, $\forall i \in \Lambda$.
Hence $\bigcup_{i\in\Lambda}\mathcal{D}_i \subseteq \bigcup_{i\in\Lambda}\mathcal{C}_i \subseteq \bigcup_{i\in\Lambda} Ncl(\mathcal{D}_i) \subseteq Ncl(\bigcup_{i\in\Lambda}\mathcal{D}_i)$.
But $\bigcup_{i\in\Lambda}\mathcal{D}_i \in N\alpha O(\mathcal{U}, \mathcal{M})$ (by proposition (3.13)). Hence $\bigcup_{i\in\Lambda}\mathcal{C}_i \in NS_{\alpha}O(\mathcal{U}, \mathcal{M})$.

Corollary 3.15:

The intersection of any family of NS_{α}-C. S. is a NS_{α}-C. S..

Proof: This follows directly from the theorem (3.14).

Remark 3.16:

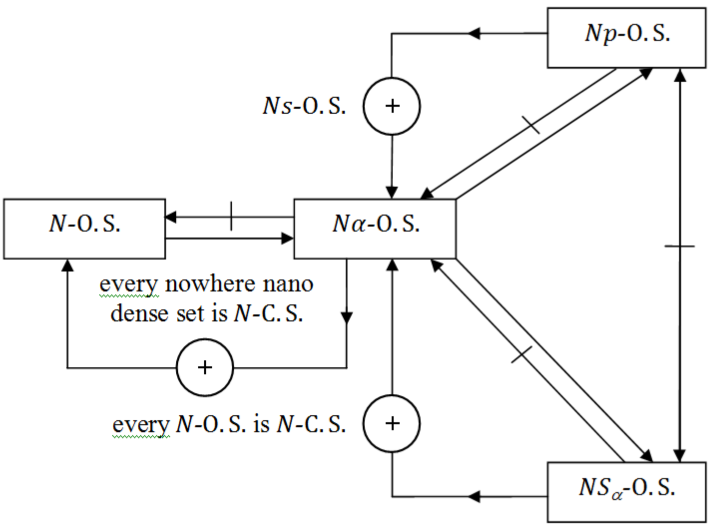
The intersection of any two NS_{α}-O. S. is not necessary NS_{α}-O. S. as in the following example.

Example 3.17:

In example (3.3), $\{p, r\}$ and $\{q, r, s\}$ are two NS_{α}-O. S., but $\{p, r\} \cap \{q, r, s\} = \{r\}$ is not NS_{α}-O. S..

Remark 3.18:

The following diagram shows the relations among the different types of weakly N-O. S. that were studied in this section:



4. NANO SEMI-α-INTERIOR AND NANO SEMI-α-CLOSURE

We present NS_α-interior and NS_α-closure and obtain some of its properties in this section.

Definition 4.1:

The union of all NS_α-O. S. in a N. T. S. $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ contained in $\mathcal{C}$ is called NS_α-interior of $\mathcal{C}$ and is denoted by $NS_\alpha int(\mathcal{C})$, $NS_\alpha int(\mathcal{C}) = \cup\{\mathcal{D}: \mathcal{D} \subseteq \mathcal{C}, \mathcal{D}$ is a NS_α-O. S.$\}$.

Definition 4.2:

The intersection of all NS_α-C. S. in a N. T. S. $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$ containing $\mathcal{C}$ is called NS_α-closure of $\mathcal{C}$ and is denoted by $NS_\alpha cl(\mathcal{C})$, $NS_\alpha cl(\mathcal{C}) = \cap\{\mathcal{D}: \mathcal{C} \subseteq \mathcal{D}, \mathcal{D}$ is a NS_α-C. S.$\}$.

Proposition 4.3:

Let $\mathcal{C}$ be any set in a N. T. S. $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, the following properties are true:
(i) $NS_\alpha int(\mathcal{C}) = \mathcal{C}$ iff $\mathcal{C}$ is a NS_α-O. S..
(ii) $NS_\alpha cl(\mathcal{C}) = \mathcal{C}$ iff $\mathcal{C}$ is a NS_α-C. S..
(iii) $NS_\alpha int(\mathcal{C})$ is the largest NS_α-O. S. contained in $\mathcal{C}$.
(iv) $NS_\alpha cl(\mathcal{C})$ is the smallest NS_α-C. S. containing $\mathcal{C}$.
Proof: (i), (ii), (iii) and (iv) are obvious.

Proposition 4.4:

Let $\mathcal{C}$ be any set in a N. T. S. $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, the following properties are true:
(i) $NS_\alpha int(\mathcal{U} - \mathcal{C}) = \mathcal{U} - (NS_\alpha cl(\mathcal{C}))$,
(ii) $NS_\alpha cl(\mathcal{U} - \mathcal{C}) = \mathcal{U} - (NS_\alpha int(\mathcal{C}))$.
Proof: (i) By definition, $NS_\alpha cl(\mathcal{C}) = \cap\{\mathcal{D}: \mathcal{C} \subseteq \mathcal{D}, \mathcal{D}$ is a NS_α-C. S.$\}$
$\mathcal{U} - (NS_\alpha cl(\mathcal{C})) = \mathcal{U} - \cap\{\mathcal{D}: \mathcal{C} \subseteq \mathcal{D}, \mathcal{D}$ is a NS_α-C. S.$\}$
$= \cup\{\mathcal{U} - \mathcal{D}: \mathcal{C} \subseteq \mathcal{D}, \mathcal{D}$ is a NS_α-C. S.$\}$
$= \cup\{\mathcal{H}: \mathcal{H} \subseteq \mathcal{U} - \mathcal{C}, \mathcal{H}$ is a NS_α-O. S.$\}$
$= NS_\alpha int(\mathcal{U} - \mathcal{C})$.

(ii) The proof is similar to (i).

Theorem 4.5:

Let $\mathcal{C}$ and $\mathcal{D}$ be two sets in a $\mathrm{N.T.S.}$ $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{M}))$. The following properties hold:
(i) $NS_{\alpha}int(\phi)=\phi$, $NS_{\alpha}int(\mathcal{U})=\mathcal{U}$.
(ii) $NS_{\alpha}int(\mathcal{C})\subseteq\mathcal{C}$.
(iii) $\mathcal{C}\subseteq\mathcal{D}\Longrightarrow NS_{\alpha}int(\mathcal{C})\subseteq NS_{\alpha}int(\mathcal{D})$.
(iv) $NS_{\alpha}int(\mathcal{C}\cap\mathcal{D})\subseteq NS_{\alpha}int(\mathcal{C})\cap NS_{\alpha}int(\mathcal{D})$.
(v) $NS_{\alpha}int(\mathcal{C})\cup NS_{\alpha}int(\mathcal{D})\subseteq NS_{\alpha}int(\mathcal{C}\cup\mathcal{D})$.
(vi) $NS_{\alpha}int(NS_{\alpha}int(\mathcal{C}))=NS_{\alpha}int(\mathcal{C})$.

Proof: (i), (ii), (iii), (iv), (v) and (vi) are obvious.

The equality in (iv) and (v) is not true in general, as the following example shows:

Example 4.6:

Let $\mathcal{U}=\{p,q,r,s\}$ with $\mathcal{U}/\mathcal{R}=\{\{q\},\{r\},\{p,s\}\}$ and $\mathcal{M}=\{p,r\}$.
Let $\tau_{\mathcal{R}}(\mathcal{M})=\{\phi,\{r\},\{p,s\},\{p,r,s\},\mathcal{U}\}$ be a $\mathrm{N.T.S.}$. The N-$\mathrm{C.S.}$ are $\mathcal{U},\{p,q,s\},\{q,r\},\{q\}$ and ϕ. The family of all $N\alpha$-$\mathrm{O.S.}$ of $\mathcal{U}$ is: $N\alpha O(\mathcal{U},\mathcal{M})=\{\phi,\{r\},\{p,s\},\{p,r,s\},\mathcal{U}\}$.
The family of all NS_{α}-$\mathrm{O.S.}$ of $\mathcal{U}$ is: $NS_{\alpha}O(\mathcal{U},\mathcal{M})=N\alpha O(\mathcal{U},\mathcal{M})\cup\{\{q,r\},\{p,q,s\}\}$.
Let $\mathcal{C}=\{q,r\},\mathcal{D}=\{p,q,s\}$. Then $NS_{\alpha}int(\mathcal{C})=\{q,r\},NS_{\alpha}int(\mathcal{D})=\{p,q,s\},\mathcal{C}\cap\mathcal{D}=\{q\}$, $NS_{\alpha}int(\mathcal{C}\cap\mathcal{D})=\phi$ and $NS_{\alpha}int(\mathcal{C})\cap NS_{\alpha}int(\mathcal{D})=\{q\}$.
It is clear that $NS_{\alpha}int(\mathcal{C})\cap NS_{\alpha}int(\mathcal{D})\nsubseteq NS_{\alpha}int(\mathcal{C}\cap\mathcal{D})$.
Let $\mathcal{C}=\{p,s\},\mathcal{D}=\{q,s\}$. Then $NS_{\alpha}int(\mathcal{C})=\{p,s\},NS_{\alpha}int(\mathcal{D})=\phi,\mathcal{C}\cup\mathcal{D}=\{p,q,s\}$, $NS_{\alpha}int(\mathcal{C}\cup\mathcal{D})=\{p,q,s\}$ and $NS_{\alpha}int(\mathcal{C})\cup NS_{\alpha}int(\mathcal{D})=\{p,s\}$.
It is clear that $NS_{\alpha}int(\mathcal{C}\cup\mathcal{D})\nsubseteq NS_{\alpha}int(\mathcal{C})\cup NS_{\alpha}int(\mathcal{D})$.

Theorem 4.7:

Let $\mathcal{C}$ and $\mathcal{D}$ be two sets in a $\mathrm{N.T.S.}$ $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{M}))$. The following properties hold:
(i) $NS_{\alpha}cl(\phi)=\phi$, $NS_{\alpha}cl(\mathcal{U})=\mathcal{U}$.
(ii) $\mathcal{C}\subseteq NS_{\alpha}cl(\mathcal{C})$.
(iii) $\mathcal{C}\subseteq\mathcal{D}\Longrightarrow NS_{\alpha}cl(\mathcal{C})\subseteq NS_{\alpha}cl(\mathcal{D})$.
(iv) $NS_{\alpha}cl(\mathcal{C}\cap\mathcal{D})\subseteq NS_{\alpha}cl(\mathcal{C})\cap NS_{\alpha}cl(\mathcal{D})$.
(v) $NS_{\alpha}cl(\mathcal{C})\cup NS_{\alpha}cl(\mathcal{D})\subseteq NS_{\alpha}cl(\mathcal{C}\cup\mathcal{D})$.
(vi) $NS_{\alpha}cl(NS_{\alpha}cl(\mathcal{C}))=NS_{\alpha}cl(\mathcal{C})$.

Proof: (i) and (ii) are evident.
(iii) By part (ii), $\mathcal{D}\subseteq NS_{\alpha}cl(\mathcal{D})$. Since $\mathcal{C}\subseteq\mathcal{D}$, we have $\mathcal{C}\subseteq NS_{\alpha}cl(\mathcal{D})$. But $NS_{\alpha}cl(\mathcal{D})$ is a NS_{α}-$\mathrm{C.S.}$. Thus Ngsg-$cl(\mathcal{D})$ is a NS_{α}-$\mathrm{C.S.}$ containing $\mathcal{C}$. Since $NS_{\alpha}cl(\mathcal{C})$ is the smallest NS_{α}-$\mathrm{C.S.}$ containing $\mathcal{C}$, we have $NS_{\alpha}cl(\mathcal{C})\subseteq NS_{\alpha}cl(\mathcal{D})$. Hence, $\mathcal{C}\subseteq\mathcal{D}\Longrightarrow NS_{\alpha}cl(\mathcal{C})\subseteq NS_{\alpha}cl(\mathcal{D})$.
(iv) We know that $\mathcal{C}\cap\mathcal{D}\subseteq\mathcal{C}$ and $\mathcal{C}\cap\mathcal{D}\subseteq\mathcal{D}$. Therefore, by part (iii), $NS_{\alpha}cl(\mathcal{C}\cap\mathcal{D})\subseteq NS_{\alpha}cl(\mathcal{C})$ and $S_{\alpha}cl(\mathcal{C}\cap\mathcal{D})\subseteq NS_{\alpha}cl(\mathcal{D})$. Hence $NS_{\alpha}cl(\mathcal{C}\cap\mathcal{D})\subseteq NS_{\alpha}cl(\mathcal{C})\cap NS_{\alpha}cl(\mathcal{D})$.
(v) Since $\mathcal{C}\subseteq\mathcal{C}\cup\mathcal{D}$ and $\mathcal{D}\subseteq\mathcal{C}\cup\mathcal{D}$, it follows from part (iii) that $NS_{\alpha}cl(\mathcal{C})\subseteq NS_{\alpha}cl(\mathcal{C}\cup\mathcal{D})$ and $NS_{\alpha}cl(\mathcal{D})\subseteq NS_{\alpha}cl(\mathcal{C}\cup\mathcal{D})$. Hence $NS_{\alpha}cl(\mathcal{C})\cup NS_{\alpha}cl(\mathcal{D})\subseteq NS_{\alpha}cl(\mathcal{C}\cup\mathcal{D})$.
(vi) Since $NS_{\alpha}cl(\mathcal{C})$ is a NS_{α}-$\mathrm{C.S.}$, we have by proposition (4.3) part (ii), $S_{\alpha}cl(NS_{\alpha}cl(\mathcal{C}))$ $=NS_{\alpha}cl(\mathcal{C})$.

The equality in (iv) and (v) is not true in general, as the following example shows:

Example 4.8:

In example (4.6), the family of all $N\alpha$-C.S. of $\mathcal{U}$ is: $N\alpha C(\mathcal{U},\mathcal{M}) = \{\mathcal{U},\{p,q,s\},\{q,r\},\{q\},\phi\}$. The family of all NS_α-C.S. of $\mathcal{U}$ is: $NS_\alpha C(\mathcal{U},\mathcal{M}) = N\alpha C(\mathcal{U},\mathcal{M})\cup\{\{p,s\},\{r\}\}$. Let $\mathcal{C}=\{p,r\},\mathcal{D}=\{q,r\}$. Then $NS_\alpha cl(\mathcal{C}) = \mathcal{U}, NS_\alpha cl(\mathcal{D}) = \{q,r\}, \mathcal{C}\cap\mathcal{D}=\{r\}$, $NS_\alpha cl(\mathcal{C}\cap\mathcal{D}) = \{r\}$ and $NS_\alpha cl(\mathcal{C})\cap NS_\alpha cl(\mathcal{D}) = \{q,r\}$. It is clear that $NS_\alpha cl(\mathcal{C})\cap NS_\alpha cl(\mathcal{D}) \nsubseteq NS_\alpha cl(\mathcal{C}\cap\mathcal{D})$.
Let $\mathcal{C}=\{p,s\},\mathcal{D}=\{r\}$. Then $NS_\alpha cl(\mathcal{C})=\{p,s\}, NS_\alpha cl(\mathcal{D})=\{r\}, \mathcal{C}\cup\mathcal{D}=\{p,r,s\}$, $NS_\alpha cl(\mathcal{C}\cup\mathcal{D})=\mathcal{U}$ and $NS_\alpha cl(\mathcal{C})\cup NS_\alpha cl(\mathcal{D}) = \{p,r,s\}$.
It is clear that $NS_\alpha cl(\mathcal{C}\cup\mathcal{D}) \nsubseteq NS_\alpha cl(\mathcal{C})\cup NS_\alpha cl(\mathcal{D})$.

Proposition 4.9:

For any subset $\mathcal{C}$ of a N.T.S. $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{M}))$, then:
(i) $Nint(\mathcal{C}) \subseteq N\alpha int(\mathcal{C}) \subseteq NS_\alpha int(\mathcal{C}) \subseteq NS_\alpha cl(\mathcal{C}) \subseteq N\alpha cl(\mathcal{C}) \subseteq Ncl(\mathcal{C})$.
(ii) $Nint(NS_\alpha int(\mathcal{C})) = NS_\alpha int(Nint(\mathcal{C})) = Nint(\mathcal{C})$.
(iii) $N\alpha int(NS_\alpha int(\mathcal{C})) = NS_\alpha int(N\alpha int(\mathcal{C})) = N\alpha int(\mathcal{C})$.
(iv) $Ncl(NS_\alpha cl(\mathcal{C})) = NS_\alpha cl(Ncl(\mathcal{C})) = Ncl(\mathcal{C})$.
(v) $N\alpha cl(NS_\alpha cl(\mathcal{C})) = NS_\alpha cl(N\alpha cl(\mathcal{C})) = N\alpha cl(\mathcal{C})$.
(vi) $NS_\alpha cl(\mathcal{C}) = \mathcal{C}\cup Nint(Ncl(Nint(Ncl(\mathcal{C}))))$.
(vii) $NS_\alpha int(\mathcal{C}) = \mathcal{C}\cap Ncl(Nint(Ncl(Nint(\mathcal{C}))))$.
(viii) $Nint(Ncl(\mathcal{C})) \subseteq NS_\alpha int(NS_\alpha cl(\mathcal{C}))$.

Proof: We shall prove only (ii), (iii), (iv), (vii) and (viii).
(ii) To prove $Nint(NS_\alpha int(\mathcal{C})) = NS_\alpha int(Nint(\mathcal{C})) = Nint(\mathcal{C})$.
Since $Nint(\mathcal{C})$ is a N-O.S., then $Nint(\mathcal{C})$ is a NS_α-O.S..
Hence $Nint(\mathcal{C}) = NS_\alpha int(Nint(\mathcal{C}))$ (by proposition (4.3)). Therefore:
$Nint(\mathcal{C}) = NS_\alpha int(Nint(\mathcal{C}))$ ....(1)
Since $Nint(\mathcal{C}) \subseteq NS_\alpha int(\mathcal{C}) \Longrightarrow Nint(Nint(\mathcal{C})) \subseteq Nint(NS_\alpha int(\mathcal{C})) \Longrightarrow$
$Nint(\mathcal{C}) \subseteq Nint(NS_\alpha int(\mathcal{C}))$.
Also, $NS_\alpha int(\mathcal{C}) \subseteq \mathcal{C} \Longrightarrow Nint(NS_\alpha int(\mathcal{C})) \subseteq Nint(\mathcal{C})$. Hence:
$Nint(\mathcal{C}) = Nint(NS_\alpha int(\mathcal{C}))$..(2)
Therefore by (1) and (2), we get $Nint(NS_\alpha int(\mathcal{C})) = NS_\alpha int(Nint(\mathcal{C})) = Nint(\mathcal{C})$.
(iii) To prove $N\alpha int(NS_\alpha int(\mathcal{C})) = NS_\alpha int(N\alpha int(\mathcal{C})) = N\alpha int(\mathcal{C})$.
Since $N\alpha int(\mathcal{C})$ is $N\alpha$-O.S., therefore $N\alpha int(\mathcal{C})$ is NS_α-O.S.. Therefore by proposition (4.3):
$N\alpha int(\mathcal{C}) = NS_\alpha int(N\alpha int(\mathcal{C}))$..(1)
Now, to prove $N\alpha int(\mathcal{C}) = N\alpha int(NS_\alpha int(\mathcal{C}))$.
Since $N\alpha int(\mathcal{C}) \subseteq NS_\alpha int(\mathcal{C}) \Longrightarrow N\alpha int(N\alpha int(\mathcal{C})) \subseteq N\alpha int(NS_\alpha int(\mathcal{C})) \Longrightarrow$
$N\alpha int(\mathcal{C}) \subseteq N\alpha int(NS_\alpha int(\mathcal{C}))$.
Also, $NS_\alpha int(\mathcal{C}) \subseteq \mathcal{C} \Longrightarrow N\alpha int(NS_\alpha int(\mathcal{C})) \subseteq N\alpha int(\mathcal{C})$. Hence:
$N\alpha int(\mathcal{C}) = N\alpha int(NS_\alpha int(\mathcal{C}))$..(2)
Therefore by (1) and (2), we get $N\alpha int(NS_\alpha int(\mathcal{C})) = NS_\alpha int(N\alpha int(\mathcal{C})) = N\alpha int(\mathcal{C})$.
(iv) To prove $Ncl(NS_\alpha cl(\mathcal{C})) = NS_\alpha cl(Ncl(\mathcal{C})) = Ncl(\mathcal{C})$.
We know that $Ncl(\mathcal{C})$ is a N-C.S., so it is NS_α-C.S.. Hence by proposition (4.3), we have:
$Ncl(\mathcal{C}) = NS_\alpha cl(Ncl(\mathcal{C}))$..(1)

To prove $\boldsymbol{Ncl(\mathcal{C})} = \boldsymbol{Ncl(NS_\alpha cl(\mathcal{C}))}$. Since $\boldsymbol{NS_\alpha cl(\mathcal{C})} \subseteq \boldsymbol{Ncl(\mathcal{C})}$ (by part (i)).
Then $\boldsymbol{Ncl(NS_\alpha cl(\mathcal{C}))} \subseteq \boldsymbol{Ncl(Ncl(\mathcal{C}))} = \boldsymbol{Ncl(\mathcal{C})} \Rightarrow \boldsymbol{Ncl(NS_\alpha cl(\mathcal{C}))} \subseteq \boldsymbol{Ncl(\mathcal{C})}$.
Since $\mathcal{C} \subseteq NS_\alpha cl(\mathcal{C}) \subseteq Ncl(NS_\alpha cl(\mathcal{C}))$, then $\mathcal{C} \subseteq Ncl(NS_\alpha cl(\mathcal{C}))$. Hence $Ncl(\mathcal{C}) \subseteq$
$Ncl\left(Ncl(NS_\alpha cl(\mathcal{C}))\right) = Ncl(NS_\alpha cl(\mathcal{C})) \Rightarrow Ncl(\mathcal{C}) \subseteq Ncl(NS_\alpha cl(\mathcal{C}))$ and therefore:
$Ncl(\mathcal{C}) = Ncl(NS_\alpha cl(\mathcal{C}))$...(2)
Now, by (1) and (2), we get that $Ncl(NS_\alpha cl(\mathcal{C})) = NS_\alpha cl(Ncl(\mathcal{C}))$.
Hence $Ncl(NS_\alpha cl(\mathcal{C})) = NS_\alpha cl(Ncl(\mathcal{C})) = Ncl(\mathcal{C})$.
(vii) To prove $NS_\alpha int(\mathcal{C}) = \mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C}))))$.
Since $NS_\alpha int(\mathcal{C}) \in NS_\alpha O(\mathcal{U}, \mathcal{M}) \Rightarrow NS_\alpha int(\mathcal{C}) \subseteq Ncl(Nint(Ncl(Nint(NS_\alpha int(\mathcal{C})))))$
$= Ncl(Nint(Ncl(Nint(\mathcal{C}))))$ (by part (ii)).
Hence $NS_\alpha int(\mathcal{C}) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$, also $NS_\alpha int(\mathcal{C}) \subseteq \mathcal{C}$. Then:
$NS_\alpha int(\mathcal{C}) \subseteq \mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C}))))$...(1)
To prove $\mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C}))))$ is a NS_α-O.S. contained in $\mathcal{C}$.
It is clear that $\mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C})))) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$ and also it is clear that $Nint(\mathcal{C}) \subseteq Ncl(Nint(\mathcal{C})) \Rightarrow Nint(Nint(\mathcal{C})) \subseteq Nint(Ncl(Nint(\mathcal{C})))$
$\Rightarrow Nint(\mathcal{C}) \subseteq Nint(Ncl(Nint(\mathcal{C}))) \Rightarrow Ncl(Nint(\mathcal{C})) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$ and
$Nint(\mathcal{C}) \subseteq Ncl(Nint(\mathcal{C})) \Rightarrow Nint(\mathcal{C}) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$ and $Nint(\mathcal{C}) \subseteq \mathcal{C}$
$\Rightarrow Nint(\mathcal{C}) \subseteq \mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C}))))$.
We get $Nint(\mathcal{C}) \subseteq \mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C})))) \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$.
Hence $\mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C}))))$ is a NS_α-O.S. (by proposition (4.3)).
Also, $\mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C}))))$ is contained in $\mathcal{C}$. Then $\mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C}))))$
$\subseteq NS_\alpha int(\mathcal{C})$ (since $NS_\alpha int(\mathcal{C})$ is the largest NS_α-O.S. contained in $\mathcal{C}$). Hence:
$\mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C})))) \subseteq NS_\alpha int(\mathcal{C})$...(2)
By (1) and (2), $NS_\alpha int(\mathcal{C}) = \mathcal{C} \cap Ncl(Nint(Ncl(Nint(\mathcal{C}))))$.
(viii) To prove that $Nint(Ncl(\mathcal{C})) \subseteq NS_\alpha int(NS_\alpha cl(\mathcal{C}))$. Since $NS_\alpha cl(\mathcal{C})$ is a NS_α-C.S., therefore $Nint(Ncl(Nint(Ncl(NS_\alpha cl(\mathcal{C}))))) \subseteq NS_\alpha cl(\mathcal{C})$ (by corollary (3.12)).
Hence $Nint(Ncl(\mathcal{C})) \subseteq Nint(Ncl(Nint(Ncl(\mathcal{C})))) \subseteq NS_\alpha cl(\mathcal{C})$ (by part (iv)).
Therefore, $NS_\alpha int\left(Nint(Ncl(\mathcal{C}))\right) \subseteq NS_\alpha int(NS_\alpha cl(\mathcal{C})) \Rightarrow$
$Nint(Ncl(\mathcal{C})) \subseteq NS_\alpha int(NS_\alpha cl(\mathcal{C}))$ (by part (ii)).

Theorem 4.10:

For any subset $\mathcal{C}$ of a N.T.S. $(\mathcal{U}, \tau_\mathcal{R}(\mathcal{M}))$. The following properties are equivalent:
(i) $\mathcal{C} \in NS_\alpha O(\mathcal{U}, \mathcal{M})$.
(ii) $\mathcal{K} \subseteq \mathcal{C} \subseteq Ncl(Nint(Ncl(\mathcal{K})))$, for some N-O.S. $\mathcal{K}$.
(iii) $\mathcal{K} \subseteq \mathcal{C} \subseteq Nsint(Ncl(\mathcal{K}))$, for some N-O.S. $\mathcal{K}$.
(iv) $\mathcal{C} \subseteq Nsint(Ncl(Nint(\mathcal{C})))$.

Proof:
$(i) \Rightarrow (ii)$ Let $\mathcal{C} \in NS_\alpha O(\mathcal{U}, \mathcal{M})$, then $\mathcal{C} \subseteq Ncl(Nint(Ncl(Nint(\mathcal{C}))))$ and $Nint(\mathcal{C}) \subseteq \mathcal{C}$.
Hence $\mathcal{K} \subseteq \mathcal{C} \subseteq Ncl(Nint(Ncl(\mathcal{K})))$, where $\mathcal{K} = Nint(\mathcal{C})$.
$(ii) \Rightarrow (iii)$ Suppose $\mathcal{K} \subseteq \mathcal{C} \subseteq Ncl(Nint(Ncl(\mathcal{K})))$, for some N-O.S. $\mathcal{K}$.
But $Nsint(Ncl(\mathcal{K})) = Ncl(Nint(Ncl(\mathcal{K})))$ (by lemma (2.6)).
Then $\mathcal{K} \subseteq \mathcal{C} \subseteq Nsint(Ncl(\mathcal{K}))$, for some N-O.S. $\mathcal{K}$.
$(iii) \Rightarrow (iv)$ Suppose that $\mathcal{K} \subseteq \mathcal{C} \subseteq Nsint(Ncl(\mathcal{K}))$, for some N-O.S. $\mathcal{K}$.
Since $\mathcal{K}$ is a N-O.S. contained in $\mathcal{C}$. Then $\mathcal{K} \subseteq Nint(\mathcal{C}) \Rightarrow Ncl(\mathcal{K}) \subseteq Ncl(Nint(\mathcal{C}))$
$\Rightarrow Nsint(Ncl(\mathcal{K})) \subseteq Nsint(Ncl(Nint(\mathcal{C})))$. But $\mathcal{C} \subseteq Nsint(Ncl(\mathcal{K}))$ (by hypothesis),

then $\mathcal{C} \subseteq Nsint(Ncl(Nint(\mathcal{C})))$.
$(iv) \Longrightarrow (i)$ Let $\mathcal{C} \subseteq Nsint(Ncl(Nint(\mathcal{C})))$.
But $Nsint(Ncl(Nint(\mathcal{C}))) = Ncl(Nint(Ncl(Nint(\mathcal{C}))))$ (by lemma (2.6)).
Hence $\subseteq Ncl(Nint(Ncl(Nint(\mathcal{C})))) \Longrightarrow \mathcal{C} \in NS_{\alpha}O(\mathcal{U}, \mathcal{M})$.

Corollary 4.11:

For any subset $\mathcal{D}$ of a N.T.S. $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{M}))$, the following properties are equivalent:
(i) $\mathcal{D} \in NS_{\alpha}C(\mathcal{U}, \mathcal{M})$.
(ii) $Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{D} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-C.S..
(iii) $Nscl(Nint(\mathcal{F})) \subseteq \mathcal{D} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-C.S..
(iv) $Nscl(Nint(Ncl(\mathcal{D}))) \subseteq \mathcal{D}$.

Proof:
$(i) \Longrightarrow (ii)$ Let $\mathcal{D} \in NS_{\alpha}C(\mathcal{U}, \mathcal{M}) \Longrightarrow Nint(Ncl(Nint(Ncl(\mathcal{D})))) \subseteq \mathcal{D}$ (by corollary (3.12)) and $\mathcal{D} \subseteq Ncl(\mathcal{D})$. Hence we get $Nint(Ncl(Nint(Ncl(\mathcal{D})))) \subseteq \mathcal{D} \subseteq Ncl(\mathcal{D})$.
Therefore $Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{D} \subseteq \mathcal{F}$, where $\mathcal{F} = Ncl(\mathcal{D})$.
$(ii) \Longrightarrow (iii)$ Let $Nint(Ncl(Nint(\mathcal{F}))) \subseteq \mathcal{D} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-C.S..
But $Nint(Ncl(Nint(\mathcal{F}))) = Nscl(Nint(\mathcal{F}))$ (by lemma (2.6)).
Hence $Nscl(Nint(\mathcal{F})) \subseteq \mathcal{D} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-C.S..
$(iii) \Longrightarrow (iv)$ Let $Nscl(Nint(\mathcal{F})) \subseteq \mathcal{D} \subseteq \mathcal{F}$, for some $\mathcal{F}$ N-C.S..
Since $\mathcal{D} \subseteq \mathcal{F}$ (by hypothesis), hence $Ncl(\mathcal{D}) \subseteq \mathcal{F} \Longrightarrow Nint(Ncl(\mathcal{D}) \subseteq Nint(\mathcal{F}) \Longrightarrow$
$Nscl(Nint(Ncl(\mathcal{D}))) \subseteq Nscl(Nint(\mathcal{F})) \subseteq \mathcal{D} \Longrightarrow Nscl(Nint(Ncl(\mathcal{D}))) \subseteq \mathcal{D}$.
$(iv) \Longrightarrow (i)$ Let $Nscl(Nint(Ncl(\mathcal{D}))) \subseteq \mathcal{D}$.
But $Nscl(Nint(Ncl(\mathcal{D}))) = Nint(Ncl(Nint(Ncl(\mathcal{D}))))$ (by lemma (2.6)).
Hence $Nint(Ncl(Nint(Ncl(\mathcal{D})))) \subseteq \mathcal{D} \Longrightarrow \mathcal{D} \in NS_{\alpha}C(\mathcal{U}, \mathcal{M})$.

5. CONCLUSION

The class of NS_{α}-O.S. defined using $N\alpha$-O.S. forms a nano topology and lay between the class of N-O.S. and the class of Ns-O.S.. The NS_{α}-O.S. can be used to derive a new decomposition of nano continuity, nano compactness, and nano connectedness.

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