ORIGINAL PAPER

ON LIFTS OF SILVER STRUCTURE

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Abstract. *In this paper, we study the complete and horizontal lift of the silver structure in the tangent bundle and prolongation of mentioned structure to tangent bundle of order* 2. *We also investigate integrability and parallelism of silver structure in tangent bundle. Moreover, we define a silver semi-Riemannian metric in tangent bundle.*

Keywords: *Silver structure; silver semi-Riemannian manifold; prolongation; tangent bundle; tangent bundle of order two; complete lift; horizontal lift; integrability.*

1. INTRODUCTION

The golden ratio is an irrational number which has an important place in the human history. We have seen this ratio in the structure of music compositions, proportions of audio frequency of harmonies, proportions of the human body, temples, statues, and fractals. Even if it is not famous as much as golden ratio, one of the irrational numbers, which glamors people, is silver ratio which is known $1+\sqrt{2}$. The value of the silver ratio is $2{,}4142135623\ldots$ and represented by θ.

The golden ratio is a number which Fibonacci numbers converge. Similarly, the silver ratio is also a number which Pell numbers converge. We shall divide into two parts a line segment such that the ratio of the sum of two-fold of the big part with the small part to the big part equals the ratio of the big part to the small part. If we take $\frac{a}{b}=x$ in the ratio $\frac{2a+b}{a}=\frac{a}{b}$, we get algebraic equation $x^2=2x+1$. The positive root of this equation is $1+\sqrt{2}$, equals to silver ratio.

In [1, 5], the geometry of golden structure which has the structure polynomial $Q(X)=X^2-X-I$ is studied. Recently, the golden structure has been studied in [2, 3, 5, 7, 8, 11, 12, 13].

In 2013, Özkan and Peltek [9] defined the silver structure, which is a new polynomial structure, by tensor field Θ of type $(1,1)$ which has structure polynomial, $Q(X)=X^2-2X-I$, on a differentiable manifold. Later, Özkan and Peltek [10] examined geometry of silver structure on a differentiable manifold.

When we exclude manifolds which are diffeomorphic to manifold M, it has the close relationship with manifold M that is total space of tangent bundle of manifold M which is represented by TM. So, we wonder whether a silver structure on manifold M has relationship with a silver structure in TM.

In this paper, firstly we introduce the silver structure. After, we study complete lifts of the silver structures in TM, and we give some examples of the silver structure in TM. Furthermore, we investigate the integrability of the silver structure, and the parallelism of the

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silver structure in TM. Next, we define silver semi-Riemannian manifold in tangent bundle. After that, we study horizontal lifts of the silver structures in tangent bundle. Finally, we study the second lift of the silver structure in tangent bundle of order two T_2M.

2. PRELIMINARIES

Throughout this paper, all manifolds, bundles, tensor fields, and connections are assumed to be of C^∞.

Definition 1 ([9,10]) *Let M be a C^∞ −differentiable manifold. A tensor field Θ of type $(1,1)$ that satisfies following equation*

$$\Theta^2 = 2\Theta + I \tag{2.1}$$

where I denotes the identity $(1,1)$ tensor field, is called a silver structure on manifold M.

Theorem 1 ([9,10]) *If P is an almost product structure on manifold M, then*

$$\Theta = I + \sqrt{2}P \tag{2.2}$$

is a silver structure on manifold M. Conversely, if Θ is a silver structure on M then

$$P = \frac{1}{\sqrt{2}}(\Theta - I) \tag{2.3}$$

is an almost product structure on manifold M.

Let the operators l and m be defined as [10]

$$\begin{aligned} l &= \frac{1}{2\sqrt{2}}\Theta - \frac{2-\theta}{2\sqrt{2}}I, \\ m &= -\frac{1}{2\sqrt{2}}\Theta + \frac{\theta}{2\sqrt{2}}I \end{aligned} \tag{2.4}$$

where $\theta = 1 + \sqrt{2}$. Then we have

$$l^2 = l, m^2 = m, lm = ml = 0, l + m = I, \tag{2.5}$$

$$l\Theta = \Theta l = \theta l, m\Theta = \Theta m = (2 - \theta)m. \tag{2.6}$$

Thus there exist in M two complementary distributions D_l and D_m corresponding to the projection tensors l and m, respectively [10].

3. THE COMPLETE LIFT OF SILVER STRUCTURE IN THE TANGENT BUNDLE

Let M be an n −dimensional differentiable manifold of class C^∞ and T_pM be tangent space at a point p of M. Then,

$$TM = \cup_{p\in M} T_pM$$

is a tangent bundle on manifold M. The tangent bundle TM is a $2n$ −dimensional differentiable manifold.

Let Θ be a tensor field of type $(1,1)$ on M. The complete lift Θ^C of Θ with local components Θ_i^h has components of the form [15]

$$\Theta^C = \begin{pmatrix} \Theta_i^h & 0 \\ \partial\Theta_i^h & \Theta_i^h \end{pmatrix}.$$

Let Θ, Φ be two tensor fields of type $(1,1)$ on M. From [15],

$$(\Theta\Phi)^C = \Theta^C\Phi^C. \tag{3.1}$$

By taking $\Theta = \Phi$ in (3.1), we have

$$(\Theta^2)^C = (\Theta^C)^2. \tag{3.2}$$

Similarly, we have $(\Theta^3)^C = (\Theta^C)^3$ by taking $\Phi = \Theta^2$ in (3.1). By repeating the same process, we obtain

$$(\Theta^k)^C = (\Theta^C)^k, k \in \mathbb{Z}^+. \tag{3.3}$$

Moreover; from [15],

$$(\Theta + \Phi)^C = \Theta^C + \Phi^C. \tag{3.4}$$

By taking complete lift of both sides of equation (2.1) and using (3.3), (3.4) and $I^C = I$, we obtain

$$(\Theta^C)^2 = 2\Theta^C + I^C. \tag{3.5}$$

Thus, we can give the following proposition.

Proposition 1 ([9,10]) *Let Θ be a tensor field of type $(1,1)$ on M. Then the complete lift Θ^C of Θ is a silver structure on TM if and only if Θ is silver structure on M.*

Let Θ be a silver structure on M. Then complete lifts l^C of l and m^C of m are complementary projection tensors on TM. Thus there exit in TM two complementary distributions D_l^C and D_m^C determined by l^C and m^C, respectively.

Proposition 2 ([9,10]) *i) If Θ is silver structure on manifold M, then the silver structure Θ^C is an isomorphism on the tangent space $T_q(TM)$ for $\forall q \in TM$.*

ii) Θ^C is invertible and its inverse $\hat{\Theta}^C = (\Theta^C)^{-1}$ satisfies following equation

$$\left(\hat{\Theta}^C\right)^2 = -2\hat{\Theta}^C + I.$$

Remark 1 *i) If T is an almost tangent structure on manifold M, then T^C and $-T^C$ are also almost tangent structures on tangent bundle TM [15].*

ii) If P is an almost product structure on manifold M, then P^C and $-P^C$ are almost product structures on tangent bundle TM [6].

ii) If J is an almost complex structure on manifold M, then J^C and $-J^C$ are almost complex structures on tangent bundle TM [15].

We can find a similar relation in a silver structure.

Proposition 3 ([9,10]) *If Θ is a silver structure on M, then $\tilde{\Theta} = 2I - \Theta^C$ is also a silver structre on TM.*

We can give the following theorem by taking complete lifts of both sides of equations (2.2), (2.3) and using Remark 1.

Theorem 2 *Let P is an almost product structure on manifold M. Then the almost product structure P^C induce a silver structure on TM as follows*

$$\Theta^C = I + \sqrt{2}P^C. \tag{3.6}$$

Conversely, let Θ is a silver structure on M. Any silver structure Θ^C yields an almost product structure on TM as following

$$P^C = \frac{1}{\sqrt{2}}(\Theta^C - I).$$

From [9, 10], we obtain the following definitions.

Let (M,T) be an almost tangent manifold. The tensor field Θ_t^C on TM defined by

$$\Theta_t^C = I + \sqrt{2}T^C$$

is called the tangent silver structure on TM. The polynomial equation satisfied by Θ_t^C is

$$(\Theta_t^C)^2 - 2\Theta_t^C + I = 0.$$

Let (M,J) be an almost complex manifold. The tensor field Θ_J^C on TM defined by

$$\Theta_J^C = I^C + \sqrt{2}J^C$$

is called the complex silver structure on TM. Θ_J^Csatisfies the equation

$$\left(\Theta_J^C\right)^2 - 2\Theta_J^C + 3I = 0.$$

From Example 2.5 of [10], we can give following example.

Example 1 *Let*

$$\Theta_{F^C} = I + \sqrt{2}F^C, \Theta_{P^C} = I + \sqrt{2}P^C, \Theta_{J^C} = I + \sqrt{2}J^C$$

where F, P are tensor fields of type $(1,1)$ on M and $J = P \circ F$. Thus we have

$$\sqrt{2}\Theta_{J^C} = \Theta_{P^C}\Theta_{F^C} - \Theta_{P^C} - \Theta_{F^C} + \theta I$$

and the triple $(\Theta_{F^C}, \Theta_{P^C}, \Theta_{J^C})$ is:

1) An almost hyperproduct (ahp)-structure on TM if and only if so is $(\Theta_F, \Theta_P, \Theta_J)$.

2) An almost biproduct complex (abpc)-structure on TM if and only if so is $(\Theta_F, \Theta_P, \Theta_J)$.

3) An almost product bicomplex (apbc)-structure on TM if and only if so is $(\Theta_F, \Theta_P, \Theta_J)$.

4) An almost hypercomplex (ahc)-structure on TM if and only if so is $(\Theta_F, \Theta_P, \Theta_J)$.

4. INTEGRABILITY CONDITIONS OF SILVER STRUCTURE IN TANGENT BUNDLE

Let Θ be a silver structure on M. Then the Nijenhuis tensor N_Θ of Θ is a tensor field of type $(1,2)$ given by [15]

$$N_\Theta(X,Y) = \Theta^2[X,Y] + [\Theta X, \Theta Y] - \Theta[\Theta X, Y] - \Theta[X, \Theta Y]$$

where $X, Y \in \Gamma(TM)$.

We have from [10] that

$$N_P(X,Y) = \frac{1}{2} N_\Theta(X,Y) \tag{4.1}$$

where N_p is Nijenhuis tensor of P almost product structure.

For $X, Y \in \Gamma(TM)$ and tensor field Θ of type $(1,1)$ on M we get [15]

$$(X+Y)^C = X^C + Y^C, [X,Y]^C = [X^C, Y^C], \Theta^C X^C = (\Theta X)^C. \tag{4.2}$$

From (2.4), (2.5), (2.6), (3.1) and (3.2), we have

$$l^C = \frac{1}{2\sqrt{2}}\Theta^C - \frac{2-\theta}{2\sqrt{2}} I, m^C = -\frac{1}{2\sqrt{2}}\Theta^C + \frac{\theta}{2\sqrt{2}} I,$$

$$l^C + m^C = I, l^C m^C = m^C l^C = 0, (l^C)^2 = l^C,\ (m^C)^2 = m^C, \tag{4.3}$$

$$\Theta^C l^C = l^C \Theta^C = \theta l^C,\ \Theta^C m^C = m^C \Theta^C = (2-\theta) m^C. \tag{4.4}$$

Let Θ, P a silver structure and an almost product structure on M, respectively. For $X, Y \in \Gamma(TM)$ the Nijenhuis tensor N_{P^C} of P^C on TM is

$$N_{P^C}(X^C, Y^C) = (P^C)^2[X^C, Y^C] + [P^C X^C, P^C Y^C] - P^C[P^C X^C, Y^C]$$
$$-P^C[X^C, P^C Y^C] \tag{4.5}$$

and the Nijenhuis tensor N_{Θ^C} of Θ^C on TM is

$$N_{\Theta^C}(X^C, Y^C) = (\Theta^C)^2[X^C, Y^C] + [\Theta^C X^C, \Theta^C Y^C] - \Theta^C[\Theta^C X^C, Y^C] - \Theta^C[X^C, \Theta^C Y^C]. \tag{4.6}$$

Proposition 4 *The complete lift D_m^C of distribution D_m in TM is integrable if and only if D_m is integrable in manifold M.*

Proof. The distribution D_m is integrable if and only if [15]

$$l[mX, mY] = 0, \tag{4.7}$$

for $X, Y \in \Gamma(TM)$. Taking complete lift of both sides of equation (4.7) and using equation (4.2), we obtain

$$l^C[m^C X^C, m^C Y^C] = 0, \tag{4.8}$$

where $X, Y \in \Gamma(TM)$. Therefore, the conditions (4.7) and (4.8) are equivalent which completes the proof.

Proposition 5 *For every $X, Y \in \Gamma(TM)$, let the distribution D_m be integrable in manifold M. That is, $lN_\Theta(mX, mY) = 0$ [10]. Then D_m^C is integrable in TM if and only if*

$$l^C N_{\Theta^C}(m^C X^C, m^C Y^C) = 0.$$

Proof. By means of (3.5), (4.4) and (4.6) we can write

$$N_{\Theta^C}(m^C X^C, m^C Y^C) = (2\theta - 2)\Theta^C[m^C X^C, m^C Y^C] + (2\theta + 2)[m^C X^C, m^C Y^C].$$

By multiplying throughout with $\frac{1}{8} l^C$ and using (4.4), we have the equation

$$\frac{1}{8} l^C N_{\Theta^C}(m^C X^C, m^C Y^C) = l^C[m^C X^C, m^C Y^C] = (lN_\Theta(mX, mY))^C.$$

If we consider $lN_\Theta(mX, mY) = 0$. Then,

$$l^C N_{\Theta^C}(m^C X^C, m^C Y^C) = 0.$$

Thus, the proof is completed.

Proposition 6 *The complete lift D_l^C of a distribution D_l in TM is integrable if and only if D_l is integrable in M.*

Proof. The distribution D_l is integrable if and only if for every $X, Y \in \Gamma(TM)$, [15]

$$m[lX, lY] = 0. \tag{4.9}$$

By taking complete lifts the both sides of equation (4.9) and using (4.2), we get

$$m^C[l^C X^C, l^C Y^C] = 0$$

where is $m^C = (I - l)^C = I - l^C$. Therefore, above two conditions are equivalent. Thus, the theorem is proved.

Proposition 7 *The distribution D_l is integrable in manifold M for $X, Y \in \Gamma(TM)$. If so $mN_\theta(lX, lY) = 0$ [10]. Then, the distribution D_l^C is integrable in TM if and only if*

$$m^C N_{\Theta^C}(l^C X^C, l^C Y^C) = 0.$$

Proof. In consequence of equations (3.2), (3.5), (4.4) and (4.6) we have

$$N_{\Theta^C}(l^C X^C, l^C Y^C) = (2 - 2\theta)\Theta^C[l^C X^C, l^C Y^C] + (2\theta + 2)[l^C X^C, l^C Y^C].$$

By multiplying with $\frac{1}{8} m^C$ throughout the above equation and using (4.4), we get

$$\frac{1}{8} m^C N_{\Theta^C}(l^C X^C, l^C Y^C) = m^C [l^C X^C, l^C Y^C] = (mN_\Theta(lX, lY))^C.$$

From $mN_\Theta(lX, lY) = 0$, we acquire the equation

$$m^C N_{\Theta^C}(l^C X^C, l^C Y^C) = 0$$

and completed proof.

Proposition 8 *For $X, Y \in \Gamma(TM)$, there is a relation between N_{P^C} and N_{Θ^C} as following*

$$N_{P^C}(X^C, Y^C) = \frac{1}{2} N_{\Theta^C}(X^C, Y^C). \tag{4.10}$$

Proof. It is obvious from (3.6), (4.1), (4.2) and (4.5).

Thus we can give the following theorem.

Proposition 9 *Let P be an almost product structure on manifold M and Θ^C be silver structure on TM. Then, Θ^C is integrable in TM if and only if P is integrable on manifold M.*

Proof. It is obvious from (4.10).

Proposition 10 *Let the silver structure Θ be integrable in manifold M. Then the silver structure Θ^C is integrable in TM if and only if $N_{\Theta^C}(X^C, Y^C) = 0$.*

Proof. From equation (4.6)

$$N_{\Theta^C}(X^C, Y^C) = (\Theta^2)^C[X^C, Y^C] + [\Theta^C X^{cC}, \Theta^C Y^C] - \Theta^C[\Theta^C X^C, Y^C] - \Theta^C[X^C, \Theta^C Y^C].$$

Since equation (4.2) and Θ is a silver structure on manifold M, we obtain

$$N_{\Theta^C}(X^C, Y^C) = (N_\Theta(X, Y))^C = 0.$$

Thus, the theorem is proved.

Both distributions D_l and D_m are integrable, if the silver structure Θ is integrable [10]. Therefore, we can give following proposition.

Proposition 11 *If Θ^C is integrable in TM then both of the distribution D_l^C and D_m^C are also integrable in TM.*

Recall that if ∇ is linear connection on M then the complete lift ∇^C of ∇ is linear connection on TM.

Let ∇ be a linear connection on manifold M. To the pair (Θ^C, ∇^C) we have two other linear connections in TM [7]

The Schouten connection

$$\tilde{\nabla}^C_{X^C} Y^C = l^c(\nabla^C_{X^C} l^C Y^C) + m^C(\nabla^C_{X^C} m^C Y^C).$$

The Vrănceanu connection

$$\breve{\nabla}^C_{X^C} Y^C = l^C(\nabla^C_{l^C X^C} l^C Y^C) + m^C(\nabla^C_{m^C X^C} m^C Y^C) + l^C[m^C X^C, l^C Y^C] \\ + m^C[l^C X^C, m^C Y^C].$$

From [10], we can say following theorems.

Proposition 12 *The projections l^C and s^C are parallel with respect to Schouten and Vrănceanu connections for every ∇^C on TM. Moreover, Θ^C is parallel with respect to Schouten and Vrănceanu connections.*

Proof. From (4.3), for $X, Y \in \Gamma(TM)$,

$$\tilde{\nabla}_{X^C} l^C Y^C = \tilde{\nabla}^C_{X^C} l^C Y^C - l^C(\tilde{\nabla}^C_{X^C} Y^C) = l^C(\nabla^C_{X^C} l^C Y^C) - l^C(\nabla^C_{X^C} l^C Y^C) = 0$$

$$\begin{aligned}(\breve{\nabla}^C_{X^C} l^C) Y^C &= \breve{\nabla}^C_{X^C} l^C Y^C - l^C(\breve{\nabla}^C_{X^C} Y^C) \\ &= l^C\left(\nabla^C_{l^C X^C} l^C Y^C\right) + l^C[m^C X^C, l^C Y^C] - l^C\left(\nabla^C_{l^C X^C} l^C Y^C\right) \\ &\quad - l^C[m^C X^C, l^C Y^C] \\ &= 0.\end{aligned}$$

The above equations can be written similarly for m^C.

Proposition 13 *The distributions D^C_l and D^C_m are parallel with respect to Schouten and Vrănceanu connections for the linear connection ∇^C on TM.*

Proof. Let $X \in \Gamma(TM)$ and $Y \in D_l$. Thus, there exist vector field X^C on TM and $Y^C \in D^C_l$. Since $m^C Y^C = (mY)^C = 0$, $l^C Y^C = (lY)^C = Y^C$

$$\tilde{\nabla}^C_{X^C} Y^C = l^C(\nabla^C_{X^C} Y^C) \in D^C_l,$$

$$\breve{\nabla}^C_{X^C} Y^C = l^C(\nabla^C_{l^C X^C} Y^C) + l^C[m^C X^C, Y^C] \in D^C_l.$$

Similar relations hold for D^C_m.

5. SILVER SEMI-RIEMANNIAN METRICS IN TANGENT BUNDLE

Let M be C^{∞} −manifold and g be semi-Riemannian metric on M. A semi-Riemannian almost product structure is a pair (g, P) where P is a almost product structure related by

$$g(PX, PY) = g(X, Y)$$

or equivalently, P is a g-symmetric endomorphism

$$g(PX, Y) = g(X, PY)$$

for every $X, Y \in \Gamma(TM)$ [4].

Proposition 14 ([15]) *Let g be a semi-Riemannian metric on M. Then g^C is a semi-Riemannian metric on TM.*

The pair (g^C, P^C) is semi-Riemannian almost product structure on TM if and only if (g, P) is a semi-Riemannian almost product structure on M. Thus, we get

$$g^C(P^C X^C, P^C Y^C) = g^C(X^C, Y^C)$$

or equivalently

$$g^C(P^C X^C, Y^C) = g^C(X^C, P^C Y^C).$$

We can give following proposition from equations (2.2) and (3.6),

Proposition 15 *The almost product structure P is a g −symmetric endomorphism if and only if the silver structure Θ^C is a g^C −symmetric endomorphism.*

Definition 2 ([10]) *A silver semi-Riemannian structure on M is a pair (g, Θ) such that*

$$g(\Theta X, Y) = g(X, \Theta Y),$$

for $X, Y \in \Gamma(TM)$. The triple (M, g, Θ) is a silver semi-Riemannian manifold.

Definition 3 *A silver semi-Riemannian structure on TM is a pair (g^C, Θ^C) such that*

$$g^C(\Theta^C X^C, Y^C) = g^C(X^C, \Theta^C Y^C)$$

for $X, Y \in \Gamma(TM)$. The triple (TM, g^C, Θ^C) is a silver semi-Riemannian manifold.

Proposition 16 *If Θ is a silver semi-Riemannian structure on M, then the complete lift Θ^C is a silver semi-Riemanian structure on TM.*

Corollary 1 *Let (M, g, Θ) be a silver semi-Riemannian manifold. Then, on the silver semi-Riemannian manifold (TM, g^C, Θ^C), we get the following:*

i) The projectors l^C, m^C are g^C −symmetric endomorphisms, i.e.,

$$g^C(l^C X^C, Y^C) = g^C(X^C, l^C Y^C),$$
$$g^C(m^C X^C, Y^C) = g^C(X^C, m^C Y^C).$$

ii) The distributions D_l^C and D_m^C are g^C −orthogonal, i.e.,

$$g^C(l^C X^C, m^C Y^C) = 0.$$

iii) The silver structure Θ^C is N_{Θ^C} −symmetric, i.e.,

$$N_{\Theta^C}(\Theta^C X^C, Y^C) = N_{\Theta^C}(X^C, \Theta^C Y^C).$$

Proposition 17 *On a locally product silver semi-Riemannian manifold, the silver structure Θ^C is integrable.*

Theorem 3 *If the linear connection*

$$\nabla^C_{X^C} Y^C = \frac{1}{4}\left[3\overline{\nabla}^C_{X^C} Y^C + \Theta^C\left(\overline{\nabla}^C_{X^C}\Theta^C Y^C\right) - \Theta^C\left(\overline{\nabla}^C_{X^C}{}^C Y^C\right) - \overline{\nabla}^C_{X^C}\Theta^C Y^C\right] + O_{P^C} Q^C(X^C, Y^C)$$

where $\overline{\nabla}^C$ is complete lift of a linear connection $\overline{\nabla}$ and Q^C is complete lift of an $(1,2)$ −tensor field Q for which $O_P Q$ is an associated Obata operator

$$O_P Q(X, Y) = \frac{1}{2}[Q(X, Y) + PQ(X, PY)]$$

for the corresponding almost product structure (2.3) then Θ^C is parallel with respect to ∇^C linear connection, i.e., $\nabla^C \Theta^C = 0$.

6. THE HORIZONTAL LIFT OF SILVER STRUCTURE

The horizontal lift L^H of a tensor field L of arbitrary type on M to TM is defined by

$$L^H = L^C - \nabla_\gamma L$$

where L is a tensor field defined by

$$L = L^{i\ldots h}_{k\ldots j} \frac{\partial}{\partial x^i} \otimes \ldots \frac{\partial}{\partial x^h} \otimes dx^k \otimes \ldots \otimes dx^j$$

in M with affine connection ∇ and $\nabla_\gamma L$ is a tensor field on TM given by

$$\nabla_\gamma L = \left(y^l \nabla_\gamma L^{i\ldots h}_{k\ldots j}\right) \frac{\partial}{\partial y^i} \otimes \ldots \otimes \frac{\partial}{\partial y^h} \otimes dx^k \otimes \ldots \otimes dx^j$$

with respect to the induced coordinates (x^h, y^h) in $\pi^{-1}(U)$ [15].

Theorem 4 *Let Θ be a tensor field of type $(1,1)$. Then the horizontal lift Θ^H of Θ is an silver structure on TM if and only if so is Θ.*

Proof. For any tensor field Θ of type $(1,1)$, we have [15]

$$(\Theta^2)^H = (\Theta^H)^2. \tag{6.1}$$

Taking the complete lift on both sides of equation (2.1), we get $(\Theta^2 - 2\Theta - I)^H = 0$. Using (6.1) and $I^H = I$, we get

$$(\Theta^H)^2 - 2\Theta^H - I = 0$$

which shows that Θ^H is a silver structure on TM.

Let Θ be a silver structure on M. Then the horizontal lift l^H of l and m^H of m are complementary projection tensors on TM. Thus there exist in TM two complementary distributions D_l^H and D_m^H determined by l^H and m^H respectively.

7. PROLONGATION OF A SILVER STRUCTURE TO TANGENT BUNDLE OF ORDER 2

Let M be an $n-$dimensional diffrentiable manifold and T_2M be the second order tangent bundle over M.

Let F be tensor field of type $(1,1)$ on manifold M. The $2nd$-lift of tensor field F is denoted F^{II}. From [14, 15], there is following relation for F, G are two tensor fields of type $(1,1)$

$$(FG)^{II} = F^{II}G^{II}.$$

Thus, we obtain

$$(F^2)^{II} = (F^{II})^2. \tag{7.1}$$

Also, there exists equation

$$(F+G)^{II} = F^{II} + G^{II}. \tag{7.2}$$

If we take $2nd$-lifts of the both sides of equation (2.1) and using (7.1), (7.2) and $I^H = I$, we get

$$(\Theta^{II})^2 = 2\Theta^{II} + I^{II}.$$

Thus, we can give the following theorem.

Theorem 5 *The second lift Θ^{II} of Θ is an silver structure on T_2M if and only if so is Θ.*

Theorem 6 *The second lift Θ^{II} of $\Theta$is integrable in T_2M if and only if Θ is integrable in M.*

Proof. It is obvious from [16, 15]

$$N^{II}(X,Y) = \big(N(X,Y)\big)^{II}$$

where N^{II} and N the Nijenhuis tensor of Θ^{II} and Θ respectively.

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