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NEW APPROACH FOR MANNHEIM CURVES IN TERMS OF INEXTENSIBLE FLOWS IN E^3

TALAT KÖRPINAR[1], MAHMUT ERGÜT[2]

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Abstract.*In this work, we study the properties of the Mannheim curves in terms of inextensible flows in E^3. Using the Frenet frame of the given curve, we present partial differential equations. We give some characterizations for curvatures of a curve in E^3.*
Keywords: *Mannheim curve, Fluid flow, Frenet frame , Partial differential equation.*

1. INTRODUCTION

Recently, fluid flow theorist has been researching fluid flows in many advances, and now fluid flow is still an important field. This fields in which fluid flow plays a party are multitudinous. Gaseous flows are figured for progression of aircraft, cars and spacecrafts, and likewise for the construction of machines such as turbines and disturbance engines. Pulpy flow investigation is necessary for seafaring operation, such as ship construction, and is extensively used in civil engineering construction such as harbour construction and coastal conservation. In chemistry, learning of fluid flow in reactor tanks is significant; in medicine, the flow in blood vessels is obtained. Various other illustrations could be submited, [1-7].

In applied differential geometry, theory of curves in space is one of the significant study areas. In the theory of curves, helices, slant helices, and rectifying curves are the most fascinating curves. One of the curves is the Mannheim curve. Space curves whose principal normals are the binormals of another curve are called Mannheim curves. The notion of Mannheim curves was discovered by A. Mannheim in 1878. These curves in Euclidean 3-space are characterised in terms of the curvature and torsion.

This study is organised as follows: Firstly, we study inextensible flows of curves in Euclidean 3-space. Secondly, using the Frenet frame of the given curve, we present partial differential equations. Finally, we give some characterizations for curvatures of a curve in Euclidean 3-space.

2. MATERIALS AND METHODS

Euclidean space E^3given by following standard flat metric

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2 ,$$

[1] Muş Alparslan University, Department of Mathematics, Muş, Turkey. E-mail:talatkorpinar@gmail.com.
[2] Namık Kemal University, Department of Mathematics, Tekirdağ, Turkey. E-mail:mergut@nku.edu.tr.

where (x_1, x_2, x_3) is a rectangular coordinate system.

The sphere of radius $r > 0$ and with center in the origin in the space E^3 is defined by

$$S^2 = \{p = (p_1, p_2, p_3) \in E^3 : \langle p, p \rangle = r^2\}.$$

$\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is denoted by the moving Frenet-Serret frame along the curve α in the space E^3. For an arbitrary curve α with first and second curvature, κ and τ in the space E^3, the following Frenet-Serret formulae are given in [2] written under matrix form

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}.$$

Definition 2.1. Let E^3 be the 3-dimensional Euclidean space with the standard inner product $\langle , \rangle$. If there exists a corresponding relationship between the space curves α and β such that, at the corresponding points of the curves, the principal normal lines of α coincides with the binormal lines of β, then α is called a Mannheim curve, and β a Mannheim partner curve of α. The pair $\{\alpha, \beta\}$ is said to be a Mannheim pair [8-13].

3. RESULTS AND DISCUSSION

Let $\alpha(u,t)$ is a one parameter family of smooth curves in E^3.

Any flow of α can be given as

$$\frac{\partial \alpha}{\partial t} = m_1 \mathbf{T} + m_2 \mathbf{N} + m_3 \mathbf{B},$$

where m_1, m_2, m_3 are differentiable functions of arclength and time.

Lemma 3.1. *Let γ be Mannheim partner of α. Then,*

$$\frac{\partial}{\partial t}\mathbf{T}^{\gamma} = [\frac{\partial}{\partial t}\cos\varphi + \sin\varphi(m_2\tau + \frac{\partial m_3}{\partial s})]\mathbf{T} + [\varepsilon\sin\varphi + \cos\varphi(m_1\kappa$$
$$-m_3\tau + \frac{\partial m_2}{\partial s})]\mathbf{N} + [\cos\varphi(m_2\tau + \frac{\partial m_3}{\partial s}) - \frac{\partial}{\partial t}\sin\varphi]\mathbf{B},$$

$$\frac{\partial}{\partial t}\mathbf{N}^{\gamma} = [\frac{\partial}{\partial t}\sin\varphi - \cos\varphi(m_2\tau + \frac{\partial m_3}{\partial s})]\mathbf{T} + [\sin\varphi(m_1\kappa - m_3\tau$$
$$+\frac{\partial m_2}{\partial s}) - \varepsilon\cos\varphi]\mathbf{N} + [\frac{\partial}{\partial t}\cos\varphi + \sin\varphi(m_2\tau + \frac{\partial m_3}{\partial s})]\mathbf{B},$$

$$\frac{\partial}{\partial t}\mathbf{B}^{\gamma} = -(m_1\kappa - m_3\tau + \frac{\partial m_2}{\partial s})\mathbf{T} + \varepsilon\mathbf{B}.$$

where

$$\varepsilon = < \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B} > .$$

Proof: From above definition we get

$$\frac{\partial}{\partial t}\mathbf{B}^{\gamma} = \frac{\partial}{\partial t}\mathbf{N} = -(m_1\kappa - m_3\tau + \frac{\partial m_2}{\partial s})\mathbf{T} + \varepsilon\mathbf{B}.$$

Using the Frenet-Serret formula, we have

$$\mathbf{T}^{\gamma} = \cos\varphi\mathbf{T} - \sin\varphi\mathbf{B}.$$

Then,

$$\frac{\partial}{\partial t}\mathbf{T}^{\gamma} = \frac{\partial}{\partial t}\cos\varphi\mathbf{T} - \frac{\partial}{\partial t}\sin\varphi\mathbf{B} + \cos\varphi\frac{\partial}{\partial t}\mathbf{T} - \sin\varphi\frac{\partial}{\partial t}\mathbf{B}.$$

Hence above equation becomes

$$\frac{\partial \mathbf{T}^{\gamma}}{\partial t} = [\frac{\partial}{\partial t}\cos\varphi + \sin\varphi(m_2\tau + \frac{\partial m_3}{\partial s})]\mathbf{T} + [\varepsilon\sin\varphi + \cos\varphi(m_1\kappa$$
$$- m_3\tau + \frac{\partial m_2}{\partial s})]\mathbf{N} + [\cos\varphi(m_2\tau + \frac{\partial m_3}{\partial s}) - \frac{\partial}{\partial t}\sin\varphi]\mathbf{B}.$$

Also,

$$\mathbf{N}^{\gamma} = \sin\varphi\mathbf{T} + \cos\varphi\mathbf{B}.$$

This implies

$$\frac{\partial}{\partial t}\mathbf{N}^{\gamma} = \frac{\partial}{\partial t}\sin\varphi\mathbf{T} + \frac{\partial}{\partial t}\cos\varphi\mathbf{B} + \sin\varphi\frac{\partial}{\partial t}\mathbf{T} + \cos\varphi\frac{\partial}{\partial t}\mathbf{B}.$$

Since

$$\frac{\partial}{\partial t}\mathbf{N}^{\gamma} = [\frac{\partial}{\partial t}\sin\varphi - \cos\varphi(m_2\tau + \frac{\partial m_3}{\partial s})]\mathbf{T} + [\sin\varphi(m_1\kappa - m_3\tau$$
$$+ \frac{\partial m_2}{\partial s}) - \varepsilon\cos\varphi]\mathbf{N} + [\frac{\partial}{\partial t}\cos\varphi + \sin\varphi(m_2\tau + \frac{\partial m_3}{\partial s})]\mathbf{B}.$$

Corollary 3.3.

$$\frac{\partial \gamma}{\partial t}=[m_1+\lambda(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]\mathbf{T}+[m_2-\frac{\partial \lambda}{\partial t}]\mathbf{N}+[m_3+\varepsilon\lambda]\mathbf{B},$$

Now we can express this lemma:

Lemma 3.4. *Let* $\frac{\partial \gamma}{\partial t}$ *be inextensible flow of* γ*. Then,*

$$\frac{\partial}{\partial s}[m_1+\lambda(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]-\kappa m_2-\frac{\partial \lambda}{\partial t}]]=0.$$

Proof: Assume that $\frac{\partial \gamma}{\partial t}$ be inextensible flow of γ. Then,

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial \gamma}{\partial t}&=[\frac{\partial}{\partial s}[m_1+\lambda(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]-\kappa m_2-\frac{\partial \lambda}{\partial t}]]\mathbf{T}\\&+[\frac{\partial}{\partial s}[m_2-\frac{\partial \lambda}{\partial t}]+\kappa m_1+\lambda(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]-\tau m_3+\varepsilon\lambda]]\mathbf{N}\\&+[\tau m_2-\frac{\partial \lambda}{\partial t}]+\frac{\partial}{\partial s}[m_3+\varepsilon\lambda]]\mathbf{B}.\end{aligned}$$

Theorem 3.5. *Let* $\frac{\partial \alpha}{\partial t}$ *be inextensible flow of* α*. If* γ *is Mannheim partner of* α*, then,*

$$\begin{aligned}&[\frac{\partial}{\partial s}[\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]-\kappa\varepsilon\sin\varphi+\cos\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]]\\&=[\sin\varphi\frac{\partial}{\partial t}(\cos\varphi\kappa^{\gamma})+(\cos\varphi\kappa^{\gamma})[\frac{\partial}{\partial t}\sin\varphi-\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]].\end{aligned}$$

Proof: Assume that γ is Mannheim partner of α.
Then, we immediately arrive at

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial}{\partial t}\mathbf{T}^{\gamma}&=[\frac{\partial}{\partial s}[\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]\\&-\kappa\varepsilon\sin\varphi+\cos\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]]\mathbf{T}\\&+[\frac{\partial}{\partial s}[\varepsilon\sin\varphi+\cos\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]\\&-\tau\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})-\frac{\partial}{\partial t}\sin\varphi]\\&+\kappa\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]]\mathbf{N}\end{aligned}$$

$$+[\frac{\partial}{\partial s}[\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})-\frac{\partial}{\partial t}\sin\varphi]$$
$$+\tau\varepsilon\sin\varphi+\cos\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]]\mathbf{B}.$$

Also, we have the following

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{T}^{\gamma}=[\sin\varphi\frac{\partial}{\partial t}(\cos\varphi\kappa^{\gamma})+(\cos\varphi\kappa^{\gamma})[\frac{\partial}{\partial t}\sin\varphi-\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]]\mathbf{T}$$
$$+(\cos\varphi\kappa^{\gamma})[\sin\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})-\varepsilon\cos\varphi]\mathbf{N}$$
$$+[\cos\varphi\frac{\partial}{\partial t}(\cos\varphi\kappa^{\gamma})+(\cos\varphi\kappa^{\gamma})[\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]]\mathbf{B}.$$

Corollary 3.6.

$$(\cos\varphi\kappa^{\gamma})[\sin\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})-\varepsilon\cos\varphi]$$
$$=[\frac{\partial}{\partial s}[\varepsilon\sin\varphi+\cos\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]-\tau\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})$$
$$-\frac{\partial}{\partial t}\sin\varphi]+\kappa\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]].$$

Corollary 3.7.

$$[\frac{\partial}{\partial s}[\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})-\frac{\partial}{\partial t}\sin\varphi]+\tau\varepsilon\sin\varphi+\cos\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]]$$
$$=[\cos\varphi\frac{\partial}{\partial t}(\cos\varphi\kappa^{\gamma})+(\cos\varphi\kappa^{\gamma})[\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]].$$

Theorem 3.8. *Let* $\frac{\partial\alpha}{\partial t}$ *be inextensible. If* γ *is Mannheim partner of* α, *then,*

$$[\frac{\partial}{\partial s}[\frac{\partial}{\partial t}\sin\varphi-\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]-\kappa\sin\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})-\varepsilon\cos\varphi]]$$

$$=-[\frac{\partial}{\partial t}(\cos\varphi\kappa^{\gamma})\cos\varphi+(\cos\varphi\kappa^{\gamma})[\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]+(\cos\varphi\tau^{\gamma})(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})].$$

Proof: By using Serret--Frenet formulas, we have

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\mathbf{N}^{\gamma}=[\frac{\partial}{\partial s}[\frac{\partial}{\partial t}\sin\varphi-\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]$$
$$-\kappa\sin\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})-\varepsilon\cos\varphi]]\mathbf{T}$$
$$+[\frac{\partial}{\partial s}[\sin\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})-\varepsilon\cos\varphi]$$
$$+\kappa\frac{\partial}{\partial t}\sin\varphi-\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]$$
$$-\tau\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]]\mathbf{N}$$
$$+[\frac{\partial}{\partial s}[\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]$$
$$+\tau\sin\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})-\varepsilon\cos\varphi]]\mathbf{B}.$$

By a direct computation, we have

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{N}^{\gamma}=-[\frac{\partial}{\partial t}(\cos\varphi\kappa^{\gamma})\cos\varphi+(\cos\varphi\kappa^{\gamma})[\frac{\partial}{\partial t}\cos\varphi$$
$$+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]+(\cos\varphi\tau^{\gamma})(m_1\kappa$$
$$-m_3\tau+\frac{\partial m_2}{\partial s})]\mathbf{T}+[\frac{\partial}{\partial t}(\cos\varphi\tau^{\gamma})-(\cos\varphi\kappa^{\gamma})$$
$$[\varepsilon\sin\varphi+\cos\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]]\mathbf{N}$$
$$+[\frac{\partial}{\partial t}(\cos\varphi\kappa^{\gamma})\sin\varphi-(\cos\varphi\kappa^{\gamma})[\cos\varphi(m_2\tau$$
$$+\frac{\partial m_3}{\partial s})-\frac{\partial}{\partial t}\sin\varphi]+\varepsilon(\cos\varphi\tau^{\gamma})]\mathbf{B}.$$

Corollary 3.9.

$$[\frac{\partial}{\partial t}(\cos\varphi\tau^{\gamma})-(\cos\varphi\kappa^{\gamma})[\varepsilon\sin\varphi+\cos\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})]]$$
$$=[\frac{\partial}{\partial s}[\sin\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})-\varepsilon\cos\varphi]+\kappa\frac{\partial}{\partial t}\sin\varphi$$
$$-\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]-\tau\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]].$$

Corollary 3.10.

$$[\frac{\partial}{\partial s}[\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]+\tau\sin\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})-\varepsilon\cos\varphi]]$$
$$=[\frac{\partial}{\partial t}(\cos\varphi\kappa^{\gamma})\sin\varphi-(\cos\varphi\kappa^{\gamma})[\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})-\frac{\partial}{\partial t}\sin\varphi]+\varepsilon(\cos\varphi\tau^{\gamma})].$$

Theorem 3.11. *Let* $\frac{\partial\alpha}{\partial t}$ *be inextensible. If* γ *is Mannheim partner of* α, *then,*

$$\frac{\partial}{\partial s}(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})=[\frac{\partial}{\partial t}(\cos\varphi\tau^{\gamma})\sin\varphi+(\cos\varphi\tau^{\gamma})[\frac{\partial}{\partial t}\sin\varphi-\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]].$$

Proof: Using Theorem 3.2, we have

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\mathbf{B}^{\gamma}=-\frac{\partial}{\partial s}(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})\mathbf{T}$$
$$-[\kappa(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})+\tau\varepsilon]\mathbf{N}+\frac{\partial\varepsilon}{\partial s}\mathbf{B}.$$

or, equivalently

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathbf{B}^{\gamma}=-[\frac{\partial}{\partial t}(\cos\varphi\tau^{\gamma})\sin\varphi+(\cos\varphi\tau^{\gamma})[\frac{\partial}{\partial t}\sin\varphi$$
$$-\cos\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]]\mathbf{T}-(\cos\varphi\tau^{\gamma})$$

$$[\sin\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})-\varepsilon\cos\varphi]\mathbf{N}$$
$$-[\frac{\partial}{\partial t}(\cos\varphi\tau^{\gamma})\cos\varphi+(\cos\varphi\tau^{\gamma})$$
$$[\frac{\partial}{\partial t}\cos\varphi+\sin\varphi(m_2\tau+\frac{\partial m_3}{\partial s})]]\mathbf{B}.$$

Corollary 3.12.

$$\kappa(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})+\tau\varepsilon]=(\cos\varphi\tau^{\gamma})[\sin\varphi(m_1\kappa-m_3\tau+\frac{\partial m_2}{\partial s})-\varepsilon\cos\varphi].$$

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