ORIGINAL PAPER

ON THE TRANSVERSAL INTERSECTION CURVE OF SPACELIKE AND TIMELIKE SURFACES IN MINKOWSKI 3-SPACE

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Abstract. *In this paper, we study the local properties of the intersection curve of a spacelike surface and a timelike surface. We derive the curvature vector, curvature and torsion for the transversal intersection for parametric-parametric intersection problem. Furthermore, we investigate some characteristic features of the intersection curve.*

Keywords: *Transversal intersection, spacelike curve, Surface-Surface intersection, Spacelike surface, Timelike surface.*

1. INTRODUCTION

At the beginning of 20th century, Albert Einstein came up with his theory of special relativity, built upon Lorentzian geometry. During the 1970s, Semi-Riemannian Geometry had become an active research area in differential geometry and its applications to a variety of subjects in mathematics and also in physics. Therefore, researchers focused on Lorentzian Geometry, the mathematical theory used in General Relativity. Since then, there has been considerable progress in the number of papers connecting differential geometry, mathematical physics and general relativity. The theory of curves and surfaces is well studied in Lorentzian Geometry analogous to Euclidean Geometry. For example, Claudel, Virbhadra and Ellis in [2] investigated the geometry of photon surfaces. Therefore, give some details and results about their geometric characterizations in 3-dimensional Minkowski Space $\mathbb{R}_1^3$. Kiehn [8] gave the result that Falaco solitons (which looks like wormholes structures in a swimming pool) can be represented as maximal surfaces in 3-dimensional Minkowski Space $\mathbb{R}_1^3$. Furthermore, there is much literature that contains differential geometry of curves and surfaces such as do Carmo [3], Struik [13]; Wilmore [15]. Unfortunately, there is less literature concerning the geometry of the intersection curve of two surfaces. In the Euclidean Geometry, Faux and Pratt [4] express the curvature of the intersection curve between two parametric surfaces. Willmore [15] provides how to obtain the unit tangent vector t, the unit principal normal vector n, and the unit binormal vector b, as well as the curvature κ and the torsion τ of an intersection curve of two implicit surfaces. Ye and Maekawa [16] describes how to compute t, n, b, κ, τ and the method for the evaluation of higher-order derivatives for both transversal and tangential intersections of all three types of intersection problems. In Global Lorentzian Geometry, Alessio and Guadalupe in [1] have already examined the local properties of a transversal intersection curve of two spacelike surfaces in $\mathbb{R}_1^3$ and given some results. The investigation in this paper is the analogue of that in [1] for the spacelike transversal intersection curve of a spacelike surface and a timelike surface. The paper is organised as

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follows. In Section 2, we present the fundamental theory of Lorentzian geometry. In Section 3, we computed the curvature and torsion of the transversal intersection spacelike curve of a spacelike surface and a timelike surface. In section 4 we give some results on the characterization of the intersection curve and in the last section, we illustrate these findings by giving some examples.

2. MATERIALS AND METHODS

In this study, the 3-dimensional Minkowski Space $\mathbb{R}_1^3$ is the pair $\left(\mathbb{R}^3, \langle , \rangle\right)$. $\mathbb{R}^3$ is a three-dimensional real vector space equipped with a Lorentz metric (inner product),

$$\begin{aligned} \langle , \rangle &: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \\ (x, y) &\to \langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3 \end{aligned} \tag{1}$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$.

A vector $x \neq 0$ in $\mathbb{R}^3$ is called spacelike, timelike or a null (lightlike), if respectively holds $\langle x, x \rangle > 0$, $\langle x, x \rangle < 0$ or $\langle x, x \rangle = 0$. Especially, the vector $x = 0$ is spacelike. If $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and its norm defined by

$$\|x\| = \left|\langle x, x \rangle\right|^{\frac{1}{2}} = \sqrt{\left|-x_1^2 + x_2^2 + x_3^2\right|} \tag{2}$$

Any given two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}_1^3$ are said to be orthogonal if $\langle x, y \rangle = 0$. A vector $x = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ which satisfies $\langle x, x \rangle = \pm 1$ is called a unit vector. Any basis $\{f_1, f_2, f_3\}$ on $\mathbb{R}_1^3$ is known as an orthogonal basis if the vectors $i = 1, 2, 3$ are mutually orthogonal vectors such that $\langle f_1, f_1 \rangle < 0$ and, $i = 2, 3$ .Therefore, for every $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}_1^3$, we have ([10])

$$x = -\langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 + \langle x, e_3 \rangle e_3 \tag{3}$$

Lemma 1. If y is a timelike vector in $\mathbb{R}_1^3$ and x is orthogonal to y then x must be a spacelike vector, [11].

Proposition 2. Let x is a spacelike vector and y is a timelike vector in $\mathbb{R}_1^3$**.** Then there is a unique called Lorentzian Timelike angle between x and y, such that

$$\langle x, y \rangle = \|x\| \|y\| \sinh \theta$$

We also define the vector product [7] or [9] of *x* and *y* (in that order) as

$$x \times y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \left(x_3 y_2 - x_2 y_3, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1\right) \tag{4}$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}_1^3$, $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ therefore, the triple scalar product of the three vectors *x*, *y*, *z* is readily given by

$$\langle z, x \times y \rangle = \begin{vmatrix} z_1 & z_2 & z_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{5}$$

where $z=(z_1,z_2,z_3)$. Remind that the vector product is nonassociative and that furthermore we have the following properties

$$\langle u\times v, x\times y\rangle = \det\begin{bmatrix}\langle u,y\rangle & \langle v,y\rangle\\ \langle u,x\rangle & \langle v,x\rangle\end{bmatrix} \tag{6}$$

where u, v, x, y are arbitrary vectors in $\mathbb{R}_1^3$ and

$$(u\times v)\times w = \langle v,w\rangle u - \langle u,w\rangle v \; ; \; u,v,w\in\mathbb{R}_1^3 \tag{7}$$

An arbitrary curve $c=c(s)$ can locally be a spacelike, timelike or null (lightlike) if all of its velocity vectors $c'(s)$ are respectively spacelike, timelike or null [12]. A non-null curve $c=c(s)$ is said to be parameterized by pseudo-arc length parameter s, if $\langle c'(s),c'(s)\rangle=\pm1$. In this case, the curve $c=c(s)$ is said to be of unit speed.

Let $c=c(s)$ be a spacelike curve parametrized by arc length s. Therefore c' is a spacelike unit vector, i.e., $\|c'\|=1$, this implies that $\langle c',c'\rangle=\|c'\|^2=1$. Then

$$\langle c',c''\rangle = 0 \tag{8}$$

According to the causal character of the vector c' we consider the following three cases(see[14]):

Case 1. $\langle c'',c''\rangle>0$

The function $\kappa(s)=\|c''(s)\|=\sqrt{\langle c''(s),c''(s)\rangle}$ is called the curvature c at s. At the points where $\kappa(s)\neq0$ a unit vector $n(s)$ in the direction $c''(s)$ is well defined by the equation

$$c''(s)=\kappa(s)n(s) \tag{9}$$

From Eq (9), we can see that the $c''(s)$ is normal to $c'(s)$. So, $n(s)$ is normal to $c'(s)$, and it is a spacelike vector and is called the normal vector at s. We shall denote by $t(s)=c'(s)$ the spacelike unit tangent vector of c at s. Thus from Eq (9). We obtain

$$t'(s)=\kappa(s)n(s) \tag{10}$$

The binormal vector

$$b(s)=t(s)\times n(s) \tag{11}$$

is the unique timelike unit vector perpendicular to the spacelike (osculating) plane $\{t(s),n(s)\}$ at every point $c(s)$ of c such that $\{t,n,b\}$ has the same orientation as $\mathbb{R}_1^3$**.** The Frenet formulas are

$$\begin{aligned} t' &= \kappa n\\ n' &= -\kappa t+\tau b\\ b' &= \tau n \end{aligned} \tag{12}$$

Case 2. $\langle c'',c''\rangle<0$

The normal vector $n(s)$ is the unit timelike vector. The binormal vector $b(s)$ is the unique spacelike unit vector perpendicular to the plane $\{t(s), n(s)\}$ at every point $c(s)$ of c such that $\{t, n, b\}$ has the same orientation as $\mathbb{R}^3_1$. The Frenet formulas are

$$\begin{aligned} t' &= \kappa n \\ n' &= \kappa t + \tau b \\ b' &= \tau n \end{aligned} \tag{13}$$

Case 3. $\langle c'', c'' \rangle = 0$

To rule out straight lines and points of inflexion on *c*, we shall assume that $c'' \neq 0$. The normal vector $n(s)$ is then the vector $c''(s)$. The binormal vector $b(s)$ is the unique null vector perpendicular to $t(s)$ at every point $c(s)$ of c such that $\langle n, b \rangle = 1$. The Frenet formulas are

$$\begin{aligned} t' &= \kappa n \\ n' &= \tau n \\ b' &= -\kappa t - \tau b \end{aligned} \tag{14}$$

Now let us evaluate the third derivative $c'''(s)$. By differentiating the equation $c'' = t' = \kappa n$ in the three cases, we obtain

Case 1. $c'''(s) = -\kappa^2 t + \kappa' n + \kappa\tau b$ (15)

Case 2. $c'''(s) = \kappa^2 t + \kappa' n + \kappa\tau b$ (16)

Case 3. $c'''(s) = \tau n$ (17)

The torsion can be computed from Eq.(15), Eq.(16) and Eq.(17) as

Case 1. $\tau = -\dfrac{\langle b, c''' \rangle}{\kappa}$ (18)

Case 2. $\tau = \dfrac{\langle b, c''' \rangle}{\kappa}$ (19)

Case 3. $\tau = \langle b, c''' \rangle$ (20)

An arbitrary plane in $\mathbb{R}^3_1$ is spacelike if the induced metric is Riemannian. Moreover, an arbitrary regular surface $X = X(u, v)$ is called a spacelike surface or timelike surface, if $X_u \times X_v$ is a timelike vector or spacelike vector respectively. The surface normal vector is perpendicular to the tangent plane and therefore at any point the unit normal vector is given by

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|} \tag{21}$$

Let $X = X(u, v)$ be a surface and $c(s) = X(u(s), v(s))$ a curve on *X*. At the point $p = c(s)$ consider the three unit vectors $t(s) = c'(s)$, $N(s)$ the normal vector to *X* at *p*, such that $\{t, U, N\}$ is a positive orthogonal frame. This orthogonal frame is called the Darboux trihedron. The unit tangent vector field t and the unit normal vector field n (in the three cases) of the curve *c* at point *p* are related by equation $t' = \kappa n$ as follows:

$$K = \frac{dt}{ds} = \kappa n = \kappa_n + \kappa_g \tag{22}$$

where κ_n is the normal curvature vector and κ_g is the geodesic vector that are the components of the curvature vector *K* of *c* in the surface normal direction and on the direction perpendicular to *T* in the surface tangent plane. So, the normal curvature vector field can be expressed as

$$\kappa_n = k_n N \tag{23}$$

where k_n is called the normal curvature of the surface at *p* in the direction *t*. In other words, k_n is the length of the projection of *K* over the normal to the surface at *p*, with a sign given by the orientation *N* on *X* at *p*. Besides, the geodesic curvature vector can be expressed as

$$\kappa_g = k_g U \tag{24}$$

where k_g is called the normal curvature of the surface at *p* in the direction *U*. In other words, k_g is the length of the projection of *K* over the normal to the surface at *p*, with a sign given by the orientation *N* on *X* at *p*.

Remark 1: Any geodesic that is a regular curve is thus characterised as a curve whose geodesic curvature $k_g = 0$ We can define the geodesic torsion τ_g of the surface *X* at *p* by using the Darboux trihedron $\{t, U, N\}$ as:

$$\tau_g = \langle U', N \rangle \tag{25}$$

Moreover, we can easily check that:

$$t' = \kappa_g U + \kappa_n N \tag{26}$$

$$U' = -\kappa_g t - \tau_g N \tag{27}$$

$$N' = \kappa_n t + \tau_g U \tag{28}$$

The equations above are analogous to Frenet-Serret formulas for the Darboux trihedron $\{t, U, N\}$.

Remark 2: A curve $c \subset X$ is a line of curvature if and only if the geodesic torsion $\tau_g = 0$

Remark 3: Let *p* be a point in *X*. An asymptotic direction of *X* at *p* is a direction on the tangent space $T_p(X)$ for which the normal curvature is zero. An asymptotic curve of *X* is a regular connected curve $c \subset X$ such that, for each p∈c, the tangent line of *c* at *p* is an asymptotic direction.

3. PROPERTIES OF A TRANSVERSAL INTERSECTION SPACELIKE CURVE OF A SPACELIKE SURFACE AND A TIMELIKE SURFACE IN $\mathbb{R}_1^3$

In this section, we compute the curvature k and torsion τ of a transversal intersection spacelike curve of a spacelike surface and a timelike surface in $\mathbb{R}_1^3$ Let $X^A = X^A(u,v)$ and $X^B = X^B(u,v)$ be a spacelike and timelike parametric surfaces, respectively. Let $c = c(s)$ be the transversal intersection spacelike curve of both surfaces X^A and X^B. This means that the spacelike tangent vector of the transversal intersection spacelike curve c lies on the tangent planes of both surfaces. Therefore, it can be obtained as the cross product of the unit surface normal vectors of the surfaces at $p = c(s)$

$$t = \frac{N^A \times N^B}{\left\| N^A \times N^B \right\|} \tag{29}$$

where N^A is the timelike unit normal vector to the spacelike surface X^A and N^B is the spacelike unit normal vector to the timelike surface X^B.

3.1. CURVATURE

Since the curvature vector c'' of the transversal intersection curve at p is perpendicular to t, must lie in the normal plane spanned by N^A and N^B. Hence, we can write it as

$$c'' = \alpha N^A + \beta N^B \tag{30}$$

where α and β are the coefficients that we need to compute. From Eq. (23) we can check that normal curvature at p in the direction t is the projection of the curvature vector $c'' = kn$ onto the timelike unit normal vector N at p. Therefore by projecting Eq. (30) onto the N^A the timelike unit normal vector of spacelike surface X^A and N^B the spacelike unit normal vector of the timelike surface, respectively. We obtain

$$\begin{aligned} k_n^A &= \alpha + \beta \sinh\theta \\ k_n^B &= \alpha \sinh\theta - \beta \end{aligned} \tag{31}$$

Since N^A and N^B are timelike and spacelike, respectively and $\sinh(\theta) = \langle N^A, N^B \rangle$, Then we have the following proposition :

Proposition 1. Let $X^A = X^A(u,v)$ and $X^B = X^B(u,v)$ be a spacelike and timelike parametric surfaces, respectively. Suppose that the $c = c(s)$ be the transversal intersection spacelike curve of both surfaces X^A and X^B, and c'' be the curvature vector. Then the curvature k of the curve c is given by

$$k^2 = \frac{\left| \left(k_n^A\right)^2 - \left(k_n^B\right)^2 + 2k_n^A k_n^B \sinh\theta \right|}{\cosh^2\theta} \tag{32}$$

Proof. Since Eq. (31) forms a linear system. By solving this system for the coefficients α and β we obtain

$$\alpha = \frac{k_n^A + k_n^B \sinh\theta}{\cosh^2\theta}, \quad \beta = \frac{-k_n^B + k_n^A \sinh\theta}{\cosh^2\theta} \tag{33}$$

By substituting Eq. (33) in Eq. (30) we have

$$c'' = \frac{k_n^A + k_n^B \sinh\theta}{\cosh^2\theta} N^A + \frac{-k_n^B + k_n^A \sinh\theta}{\cosh^2\theta} N^B \tag{34}$$

Now by using the same manner Ye and Maekawa [16] and Alessio and Guadalupe [1], we can determine the two normal curvatures k_n^A and k_n^B at p and therefore we compute the curvature vector from Eq.(34). Finally, the curvature of the intersection spacelike curve c at p can be computed by using Eq.(9), and Eq. (34) as follows

$$\kappa^2 = \left|\langle c'', c''\rangle\right|$$

$$= \left| \left(\frac{k_n^A + k_n^B \sinh\theta}{\cosh^2\theta} \right)^2 \langle N^A, N^A \rangle + \left(\frac{-k_n^B + k_n^A \sinh\theta}{\cosh^2\theta} \right)^2 \langle N^B, N^B \rangle + 2 \left(\frac{k_n^A + k_n^B \sinh\theta}{\cosh^2\theta} \right) \left(\frac{-k_n^B + k_n^A \sinh\theta}{\cosh^2\theta} \right) \langle N^A, N^B \rangle \right|$$

$$= \frac{\left| \left(k_n^A\right)^2 - \left(k_n^B\right)^2 + 2k_n^A k_n^B \sinh\theta \right|}{\cosh^2\theta}$$

3.2. TORSION

Since the timelike unit normal vector N^A and spacelike unit normal vector N^B lie in the normal plane, the term $k'n + k\tau b$ in Eq.(15) and Eq.(16) and the term τn in Eq.(17) can be replaced by $\gamma N^A + \delta N^B$. Thus

Case 1. $c'''(s) = -\kappa^2 t + \gamma N^A + \delta N^B$ (35)

Case 2. $c'''(s) = \kappa^2 t + \gamma N^A + \delta N^B$ (36)

Case 3. $c'''(s) = \gamma N^A + \delta N^B$ (37)

Now, if we projected $c'''(s)$ onto N^A the timelike unit normal vector of spacelike surface X^A and N^B the spacelike unit normal vector of timelike surface X^B and denoted by λ_n^A and λ_n^B respectively. We obtain

$$\begin{aligned} \lambda_n^A &= \gamma + \delta \sinh\theta \\ \lambda_n^B &= \gamma \sinh\theta - \delta \end{aligned} \tag{38}$$

Thus, we can give the following

Proposition 2. Let $X^A = X^A(u,v)$ and $X^B = X^B(u,v)$ be a spacelike and timelike parametric surfaces, respectively. Suppose that the $c = c(s)$ is the transversal intersection spacelike curve of both surfaces X^A and X^B. Therefore c'' is a spacelike, timelike or null vector. Then the torsion of the curve c is given by

Case 1. $\tau = -\dfrac{\langle b, c'''\rangle}{\kappa}$

$$= -\frac{1}{\kappa \cosh^2\theta} \left[\left(\frac{\lambda_n^A + \lambda_n^B \sinh\theta}{\cosh^2\theta} \right) \langle b, N^A \rangle + \left(\frac{-\lambda_n^B + \lambda_n^A \sinh\theta}{\cosh^2\theta} \right) \langle b, N^B \rangle \right] \tag{39}$$

Case 2. $\tau = \dfrac{\langle b, c'''\rangle}{\kappa}$

$$= \frac{1}{\kappa \cosh^2\theta} \left[\left(\frac{\lambda_n^A + \lambda_n^B \sinh\theta}{\cosh^2\theta} \right) \langle b, N^A \rangle + \left(\frac{-\lambda_n^B + \lambda_n^A \sinh\theta}{\cosh^2\theta} \right) \langle b, N^B \rangle \right] \tag{40}$$

Case 3. $\tau = \langle b, c'''\rangle$ (41)

$$=\frac{1}{\cosh^2\theta}\left[\left(\frac{\lambda_n^A+\lambda_n^B\sinh\theta}{\cosh^2\theta}\right)\langle b,N^A\rangle+\left(\frac{-\lambda_n^B+\lambda_n^A\sinh\theta}{\cosh^2\theta}\right)\langle b,N^B\rangle\right]$$

where the binormal vector b is evaluated in the three cases, and the curvature k is computed by Eq.(32).

Proof. By solving the coefficients γ and δ from linear system Eq (38) we have

$$\gamma=\frac{\lambda_n^A+\lambda_n^B\sinh\theta}{\cosh^2\theta},\ \delta=\frac{-\lambda_n^B+\lambda_n^A\sinh\theta}{\cosh^2\theta} \tag{42}$$

Moreover, by substituting in Eq. (35), (36), (37) we obtain

Case 1.

$$c'''(s)=-\kappa^2 t+\left(\frac{\lambda_n^A+\lambda_n^B\sinh\theta}{\cosh^2\theta}\right)N^A+\left(\frac{-\lambda_n^B+\lambda_n^A\sinh\theta}{\cosh^2\theta}\right)N^B \tag{43}$$

Case 2.

$$c'''(s)=\kappa^2 t+\left(\frac{\lambda_n^A+\lambda_n^B\sinh\theta}{\cosh^2\theta}\right)N^A+\left(\frac{-\lambda_n^B+\lambda_n^A\sinh\theta}{\cosh^2\theta}\right)N^B \tag{44}$$

Case 3.

$$c'''(s)=\left(\frac{\lambda_n^A+\lambda_n^B\sinh\theta}{\cosh^2\theta}\right)N^A+\left(\frac{-\lambda_n^B+\lambda_n^A\sinh\theta}{\cosh^2\theta}\right)N^B \tag{45}$$

Now by using the same methods as the ones of Ye and Maekawa [16] and Alessio and Guadalupe, [1]. We can compute λ_n^A and λ_n^B to evaluate c'''. As a result, from Eq.(18), (19) and (20) and the Eq.(43), (44),(45), we can easily obtain Eq.(39), (40) and (41).

4. CHARACTERIZATION OF TRANSVERSAL SPACELIKE CURVE OF A SPACELIKE SURFACE AND A TIMELIKE SURFACE

In this section, we obtain two characterizations of transversal intersection spacelike curves of a spacelike surface X^A and a timelike surface X^B that are given by Theorem 1 and Theorem 2. Let $\{T,U^A,N^A\}$ and $\{T,U^B,N^B\}$ be positive orthogonal frames on X^A and X^B, respectively, where $U^A=N^A\times T$ and $U^B=N^B\times T$.

Lemma 2. Suppose that the $c=c(s)$ is the transversal intersection spacelike curve of spacelike surface X^A and timelike surface X^B. If k_g^A and k_g^B are the geodesic curvatures of X^A and X^B, respectively, then we have

$$k_g^A=\frac{k_n^B-k_n^A\sinh\theta}{\cosh\theta} \tag{46}$$

$$k_g^B=\frac{k_n^A+k_n^B\sinh\theta}{\cosh\theta} \tag{47}$$

where k_n^A and k_n^B are the normal curvatures of X^A and X^B, respectively.

Proof. Using Eq. (7) Moreover, Prop. (2) we have that

$$U^A = N^A \times T = \frac{N^A \times N^A \times N^B}{\left\| N^A \times N^B \right\|} = \frac{\left\langle N^A, N^A \right\rangle N^B - \left\langle N^A, N^B \right\rangle N^A}{\left\| N^A \times N^B \right\|}$$

$$= \frac{N^B - \sinh\theta N^A}{\cosh\theta} \tag{48}$$

similarly, we have

$$U^B = N^B \times T = \frac{N^B \times N^A \times N^B}{\left\| N^A \times N^B \right\|} = \frac{\left\langle N^A, N^B \right\rangle N^B - \left\langle N^B, N^B \right\rangle N^A}{\left\| N^A \times N^B \right\|}$$

$$= \frac{\sinh\theta N^B + N^A}{\cosh\theta} \tag{49}$$

From Eq (34)

$$c'' = \frac{k_n^A + k_n^B \sinh\theta}{\cosh^2\theta} N^A + \frac{-k_n^B + k_n^A \sinh\theta}{\cosh^2\theta} N^B$$

Thus Using Eq. (22), (23), (24) and (48) we obtain Eq.(46) and (47). Indeed,

$$k_g^A = \left\langle c'', U^A \right\rangle$$

$$= \left\langle \frac{k_n^A + k_n^B \sinh\theta}{\cosh^2\theta} N^A + \frac{-k_n^B + k_n^A \sinh\theta}{\cosh^2\theta} N^B, U^A \right\rangle$$

$$= \left\langle \frac{k_n^A + k_n^B \sinh\theta}{\cosh^2\theta} N^A + \frac{-k_n^B + k_n^A \sinh\theta}{\cosh^2\theta} N^B, \frac{N^B - \sinh\theta N^A}{\cosh\theta} \right\rangle$$

$$= \frac{k_n^B - k_n^A \sinh\theta}{\cosh\theta}$$

similarly, we obtain

$$k_g^B = \left\langle c'', U^B \right\rangle = \frac{k_n^A + k_n^B \sinh\theta}{\cosh\theta}$$

Lemma 3. Suppose that spacelike surface X^A and timelike surface X^B intersect along a spacelike curve $c = c(s)$ and let θ be the angle made by the normal vectors of X^A and X^B at $p \in c$. Then we have

$$\frac{d\theta}{ds} = \tau_g^B - \tau_g^A \tag{50}$$

where τ_g^A and τ_g^B are the geodesic torsions of X^A and X^B, respectively.

Proof. Applying Eq. (27) to the surfaces X^A *and* X^B we have

$$\left(N^A\right)' = k_n^A T + \tau_g^A U^A \tag{51}$$

and

$$\left(N^B\right)' = k_n^B T + \tau_g^B U^B \tag{52}$$

Now, differentiating $\left\langle N^A, N^B \right\rangle = \sinh\theta$ and using Eq. (51), (52), (48) and (49) we obtain

$$\cosh\theta \frac{d\theta}{ds} = \left\langle \left(N^A\right)', N^B \right\rangle + \left\langle N^A, \left(N^B\right)' \right\rangle$$

$$= \tau_g^A \left\langle U^A, N^B \right\rangle + \tau_g^B \left\langle U^B, N^A \right\rangle$$

$$
\begin{aligned}
&= \tau_g^A \left\langle \frac{N^B - \sinh\theta N^A}{\cosh\theta}, N^B \right\rangle + \tau_g^B \left\langle \frac{\sinh\theta N^B + N^A}{\cosh\theta}, N^A \right\rangle \\
&= \left(\tau_g^B - \tau_g^A\right)\cosh\theta
\end{aligned}
\tag{53}
$$

So we have

$$\frac{d\theta}{ds} = \tau_g^B - \tau_g^A$$

Theorem 1. Suppose that a spacelike surface X^A and a timelike surface X^B intersect along a spacelike curve $c = c(s)$. Then c is a geodesic curve of X^A and X^B if and only if c is an asymptotic curve of X^A and X^B (i.e. $k_n^A = k_n^B = 0$ where k_n^A and k_n^B are the normal curvatures of X^A and X^B, respectively).

Proof. If c is a geodesic curve of X^A and X^B from Remark (1) it follows that $k_g^A = k_g^B = 0$ Now, using Eq.(46) and Eq.(47) from Lemma (2), we have

$$
\begin{cases}
k_n^B - k_n^A \sinh\theta = 0 \\
k_n^A + k_n^B \sinh\theta = 0
\end{cases}
\tag{54}
$$

Solving the homogeneous linear system Eq.(54), we obtain $k_n^A = k_n^B = 0$, since

$$\det\begin{pmatrix} -\sinh\theta & 1 \\ 1 & \sinh\theta \end{pmatrix} = -\sinh^2\theta - 1 = -\cosh^2\theta \neq 0$$

Moreover, therefore, c is an asymptotic curve of X^A *and* X^B. Conversely, if $k_n^A = k_n^B = 0$ then from Eq.(46) and Eq.(47) it follows that $k_g^A = k_g^B = 0$ and, therefore, c is a geodesic curve of X^A *and* X^B.

Theorem 2. Suppose that spacelike surface X^A and timelike surface X^B intersect along a spacelike curve $c = c(s)$ and let θ be the angle made by the normal vectors of X^A and X^B at $p \in c$. Assume that c is a line of curvature of X^A. Then θ is constant if and only if c is a line of curvature of X^B.

Proof. If c is a line of curvature of X^A from Remark (2), it follows that $\tau_g^A = 0$. Using Eq.(50) from the above Lemma (3) and the fact that θ is constant we have $\tau_g^B = 0$, and this implies that c is a line of curvature of X^B Conversely, if c is a line of curvature of X^B then we obtain $\tau_g^A = \tau_g^B = 0$ and therefore from Eq.(50) it follows that $\frac{d\theta}{ds} = \tau_g^B - \tau_g^A = 0 - 0 = 0$, and this implies that θ is constant.

5. EXAMPLES

To illustrate Proposition 1, Proposition 2, Theorem 1 and Theorem 2, we present now some examples.

Example 1. The parametric surface X^A is a Hyperbolical cylinder given by $X^A(u,v)=(\cosh u, \sinh u, v)$. Moreover, parametric surface X^B is a plane given by $X^B(r,w)=(r,3,w)$

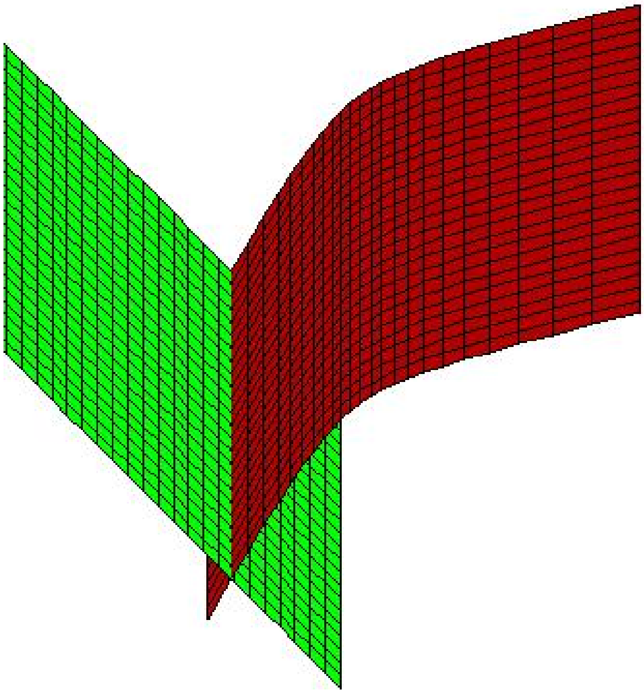

Figure 1. Intersection of a Hyperbolic cylinder with a plane.

The intersection curve is a straight line. It is both an asymptotic curve and a geodesic of X^A and X^B. Indeed, since the intersection curve is a straight line, the curvature is $k=0$, then $k_n^A=k_n^B=0$ and $k_g^A=k_g^B=0$.

Example 2. The parametric surface X^A is a Hyperbolic cylinder given by $X^A(u,v)=(\cosh u, \sinh u, v)$

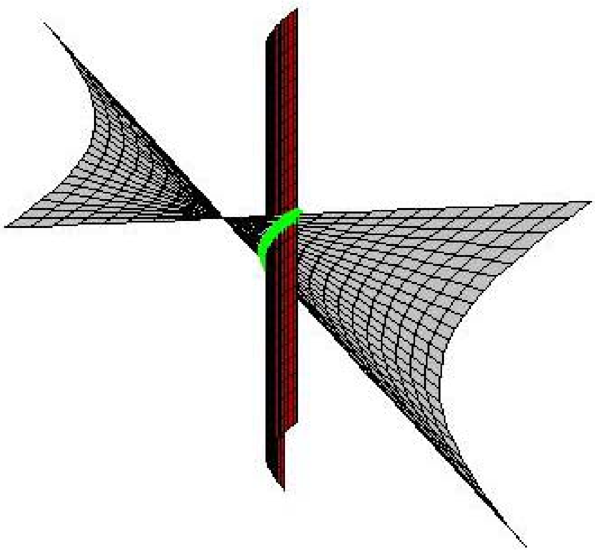

Figure 2. Intersection of a Hyrperbolic cylinder with a ruled surface.

Moreover, parametric surface X^B is ruled surface given by

$$X(r,w) = c(r) + w\beta(r)$$

where $c(r) = (\cosh r, \sinh r, 0)$ and $\beta(r) = \left(2N^A + \frac{1}{2}U^A\right)$. The normal vector $N^A(c(r)) = (-\cosh r, -\sinh r, 0)$ and $U^A(c(r)) = (0,0,-1)$. The $c(r)$ is the intersection curve. The derivative of the normal vector is $\frac{dN^A(c(r))}{dr} = (-\sinh r, -\cosh r, 0)$ and $U^A(c(r)) = (0,0,-1)$ then $\tau_g^A = \left\langle \left(N^A(c(r))\right)', U^A \right\rangle = 0$ from remark (2) it follows that c is a line of curvature in X^A The derivative of the normal vector is $\frac{dN^B(c(r))}{dr} = \frac{1}{\sqrt{3}}(\sinh r, \cosh r, 0)$ and $U^B(c(r)) = \frac{1}{\sqrt{3}}(4\cosh r, 4\sinh r, 1)$ $\tau_g^B = \left\langle \left(N^B(c(r))\right)', U^B \right\rangle = 0$ From remark (2) it follows that c is a line of curvature in X^B The angle made by the normal vectors of X^A and X^B is constant. Indeed since $N^A(c(r)) = (-\cosh r, -\sinh r, 0)$ and $N^B(c(r)) = \frac{1}{\sqrt{3}}(\cosh r, \sinh r, 2)$ $\left\langle N^A(c(r)), N^B(c(r)) \right\rangle = \frac{1}{\sqrt{3}}\left(\cosh^2(r) - \sinh^2(r)\right) = \frac{1}{\sqrt{3}}$.

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