ORIGINAL PAPER

ON THE NUMERICAL SOLUTION OF WHITHAM-BROER-KAUP SHALLOW WATER WAVE EQUATIONS

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Abstract. *The variational iteration method (VIM) is applied to find the approximate solutions of the Whitham-Broer-Kaup (WKB) shallow water model. The results obtained are compared with the exact solutions. The results show that the variational iteration method is efficient and practically well suited for solving coupled nonlinear water wave equations.*
Keywords: *variational iteration method, Whitham-Broer-Kaup equations, coupled system.*

1. INTRODUCTION

Nonlinear system of partial differential equations plays important roles in many physical phenomena that appear in many engineering and sciences such as fluid mechanics, solid state and plasma physics, chemical kinetics and mathematical biology.

The study of coupled nonlinear system of partial differential equations had attracted the attention of various researchers in order to find the best numerical solutions of the equations.

In this research, the coupled Whitham-Broer-Kamp (WBK) [1], Broer [2] and Kamp [3] are considered. The equations model the propagation of shallow water waves, with dispersion relations. The WBK equations are of the form:

$$\begin{aligned} &u_t + uu_x + v_x + \beta u_{xx} = 0 \\ &v_t + uv_x + vu_x + \alpha u_{xxx} - \beta v_{xx} = 0 \end{aligned} \tag{1}$$

where $u = u(x,t)$ is the horizontal velocity, $v = v(x,t)$ is the height that deviates from equilibrium position of the liquid, α and β are constants which are represented diffusion powers [4].

The equations described dispersive waves. It represents the modified Boussinesq (MB) equations when $\alpha = 1$ and $\beta = 0$. If $\alpha = 0$ and $\beta \neq 0$, the system models the classical long-wave equations that describe shallow water wave with dispersion.

The exact solution of $u(x,t)$ and $v(x,t)$ are given by [5] as:

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$$u(x,t) = \lambda - 2k\left(\alpha + \beta^2\right)^{0.5} \coth\left[k\left(x + x_\circ\right) - \lambda t\right]$$
$$v(x,t) = \lambda - 2k^2\left(\alpha + \beta^2 + \beta\left(\alpha + \beta^2\right)^{0.5}\right)\csc h^2\left[k\left(x + x_\circ\right) - \lambda t\right] \tag{2}$$

where λ, k *and* $x_\circ$ are arbitrary constants. Some researchers have used different methods to solve the WBK equations. In [6], the Laplace decomposition method and the pade approximation were used to solve WBK equations. Also in [7] the homotopy analysis method was employed to find the approximate travelling wave solutions of coupled WBK shallow water equation. Jamshad Ahmed et. al [8] provided the exact solution of WBK shallow water wave equations using He's polynomial and Adomain decomposition method.

The aim of this research article is to apply the variational iteration method to solve the Whitham-Broer-Kamp shallow water wave equations. Inokuti et. al [9] proposed a general Lagrange multiplier to solve problems in quantum mechanics, later, He [10-11] modified it into an iterative method that is called variational iteration method. This method is capable of greatly reducing the computational time while still maintaining high efficiency [10-16].

2. BASIC CONCEPTS OF VARIATIONAL ITERATION METHOD

The basic idea of the He's Variational Iteration Method (VIM) [10-16], can be explained by considering the following nonlinear partial differential equations

$$Lu + Nu = g(x) \tag{3}$$

where L is the linear operator, N is the nonlinear operator and $g(x)$ is the inhomogeneous term. According to the method, we can construct a correction functional.

The corresponding variational iteration method for solving (3) is given as

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\xi)\left[Lu_n(\xi) + N\breve{u}_n(\xi) - g(\xi)\right]d\xi, \tag{4}$$

where λ is a Lagrange multiplier which can be identified optimally by variational iteration method. The subscript n denote the nth approximation, $\bar{u}_n$ is considered as a restricted variation i.e $\delta\bar{u}_n = 0$. The successive approximation $u_{n+1,} n \geq 0$ of the solution u can be easily obtained by determine the Lagrange multiplier and the initial guess u_0, consequently, the solution is given by $u = \lim_{n\to\infty} u_n$.

3. APPLICATION

In this section, the Variational iteration method is used to solve the coupled Whitham-Broer-Kamp equations (1) with different parameters.

Example 3.1: Consider the WBK (1) with $\alpha = 0$ and $\beta = 0.5$ then the equation becomes:

$$\begin{aligned} & u_t + uu_x + v_x + 0.5u_{xx} = 0 \\ & v_t + uv_x + vu_x - 0.5v_{xx} = 0 \end{aligned} \qquad (5)$$

Subject to the following conditions:

$$\begin{aligned} & u(x,0) = \lambda - k \coth[k(x + x_\circ)] \\ & v(x,0) = -k^2 \operatorname{csc}h^2 [k(x + x_\circ)] \end{aligned} \qquad (6)$$

According to the VIM, the correction functional (4) of equation (6) as:

$$\begin{aligned} & u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(x,\xi)\left[u_n(x,\xi)_\xi + u_n(x,\xi)\breve{u}_n(x,\xi)_x + v_n(x,\xi) + 0.5\breve{u}_n(x,\xi)_{xx}\right]d\xi \\ & (x,t) = v_n(x,t) + \int_0^t \lambda(x,\xi)\left[u_n(x,\xi)_\xi + u_n(x,\xi)\breve{v}_n(x,\xi)_x + v_n(x,\xi)\breve{u}_n(x,\xi)_x - 0.5\breve{v}_n(x,\xi)_{xx}\right]d\xi \end{aligned} \qquad (7)$$

where $\lambda(x,\xi) = -1$ can be found from equation (7). Hence, the correction functional as shown in equation (7) becomes the iteration formula as follows:

$$\begin{aligned} & u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left[u_n(x,\xi)_\xi + u_n(x,\xi)u_n(x,\xi)_x + v_n(x,\xi) + 0.5u_n(x,\xi)_{xx}\right]d\xi \\ & V_{n+1}(x,t) = v_n(x,t) - \int_0^t \left[u_n(x,\xi)_\xi + u_n(x,\xi)v_n(x,\xi)_x + v_n(x,\xi)u_n(x,\xi)_x - 0.5v_n(x,\xi)_{xx}\right]d\xi \end{aligned} \qquad (8)$$

Using the above iteration, the following approximation can be obtained.

$$\begin{aligned} & u_0(x,t) = \lambda - k \coth(k(x + x_0)) \\ & v_0(x,t) = -k^2 \operatorname{csc}h^2 (k(x + x_0)) \end{aligned} \qquad (9)$$

$$\begin{aligned} & u_1(x,t) = \lambda - 1.k \coth(k(x + x_0)) + 1.k^2 t\lambda - 1.k^2 t\lambda \coth(k(x + x_0))^2 - 2.k^3 t \coth(k(x + x_0)) \\ & + 2.k^3 t \coth(k(x + x_0))^3 - 2.k^3 \operatorname{csc}h(k(x + x_0))^2 \coth(k(x + x_0))t \\ & (k(x + x_0))^3 \operatorname{csc}h(k(x + x_0))^2 + 1.333333333t^3 \coth(k(x + x_0)) \\ & \operatorname{csc}h(k(x + x_0))^2 k^7 \end{aligned} \qquad (10)$$

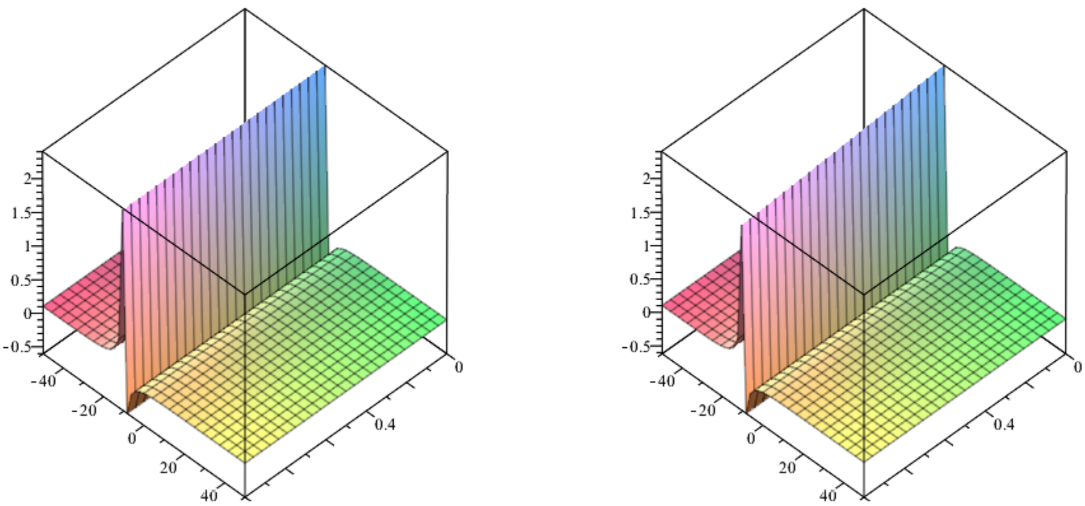
$$\begin{aligned} & v_1(x,t) = -1.k^2 \operatorname{csc}h(k(x + x_0))^2 - 2.k^3 \operatorname{csc}h(k(x + x_0))^2 \coth(k(x + x_0))t\lambda + 3.k^4 \operatorname{csc}h(k(x + x_0))^2 \\ & \coth(k(x + x_0))^2 t - 1.k^4 \operatorname{csc}h(k(x + x_0))^2 t + 1.k^3 t \coth(k(x + x_0)) - 1.k^3 t \coth(k(x + x_0))^3 \\ & \operatorname{csc}h(k(x + x_0))^2 - 5.333333333t^3 k^7 \coth \end{aligned}$$

$$u_2(x,t)=\lambda-1.k\coth(k(x+x_0))-2.k^3\operatorname{csc}h(k(x+x_0))^2\coth(k(x+x_0))t-1.k^2t\lambda\coth(k(x+x_0))^2$$
$$+7.333333333t^3k^6\coth(k(x+x_0))^4\lambda-3.333333333t^3k^6\coth(k(x+x_0))^6\lambda-7.999999999t^3k^7$$
$$\coth(k(x+x_0))^5\operatorname{csc}h(k(x+x_0))^2+10.66666667t^3k^7\coth(k(x+x_0))^3\operatorname{csc}h(k(x+x_0))^2$$
$$-2.666666666t^3\coth(k(x+x_0))\operatorname{csc}h(k(x+x_0))^2k^7+4.000000000t^3\coth(k(x+x_0))^3\operatorname{csc}h(k(x+x_0))^4k^7$$
$$-1.333333333t^3\coth(k(x+x_0))\operatorname{csc}h(k(x+x_0))^4k^7-1.333333333t^3k^5\lambda^2\coth(k(x+x_0))^3$$
$$+0.6666666666t^3k^5\lambda^2\coth(k(x+x_0))^5-4.666666666t^3k^6\lambda\coth(k(x+x_0))^2+0.6666666666t^3k^5\lambda^2$$
$$\coth(k(x+x_0))+0.6666666666t^3k^6\lambda\operatorname{csc}h(k(x+x_0))^2+16.00000000t^2k^5\operatorname{csc}h(k(x+x_0))^2\coth(k(x+x_0))^3$$
$$-10.00000000t^2k^5\operatorname{csc}h(k(x+x_0))^2\coth(k(x+x_0))+1.000000000t^2k^3\lambda^2\coth(k(x+x_0))$$
$$-1.000000000t^2k^3\lambda^2\coth(k(x+x_0))^3-8.000000000t^2\lambda k^4\coth(k(x+x_0))^2+6.000000000t^2\lambda k^4$$
$$\coth(k(x+x_0))^4+2.000000000t^2\lambda k^4\operatorname{csc}h(k(x+x_0))^2+1.k^2t\lambda-2.k^3t\coth(k(x+2x_0))+2.k^3t\coth(k(x+x_0))^3$$
$$+0.6666666666t^3k^6\lambda-0.5000000000t^2k^4-4.000000000t^3\coth(k(x+x_0))^2\operatorname{csc}h(k(x+x_0))^2k^6\lambda$$
$$+3.333333333t^3\coth(k(x+x_0))^4\operatorname{csc}h(k(x+x_0))^2k^6\lambda-6.000000000t^2k^4\operatorname{csc}h(k(x+x_0))^2\coth(k(x+x_0))^2\lambda$$
$$+2.000000000t^2\lambda k^4+6.666666666t^3k^7\coth(k(x+x_0))^3-9.333333332t^3k^7\coth(k(x+x_0))^5$$
$$4.000000000t^3k^7\coth(k(x+x_0))^7-1.333333333t^3k^7\coth(k(x+x_0))+2.000000000t^2k^4\coth(k(x+x_0))^2$$
$$-1.500000000t^2k^4\coth(k(x+x_0))^4-6.000000000t^2k^5\coth(k(x+x_0))+16.00000000t^2k^5\coth(k(x+x_0))^3$$
$$-10.00000000t^2k^5\coth(k(x+x_0))^5$$

$$v_2(x,t)=-2.k^3\cosh(k(x+x_0))^2\coth(k(x+x_0))t\lambda-1.k^2\operatorname{csc}h(k(x+x_0))^2+3.k^4\operatorname{csc}h(k(x+x_0))^2\coth(k(x+x_0))^2t$$
$$-3.666666666t^3k^6\coth(k(x+x_0))^4\lambda+1.666666666t^3k^6\coth(k(x+x_0))^6\lambda+4.00000000t^3k^7\coth(k(x+x_0))^5$$
$$\operatorname{csc}h(k(x+x_0))^2-5.333333333t^3k^7\coth$$

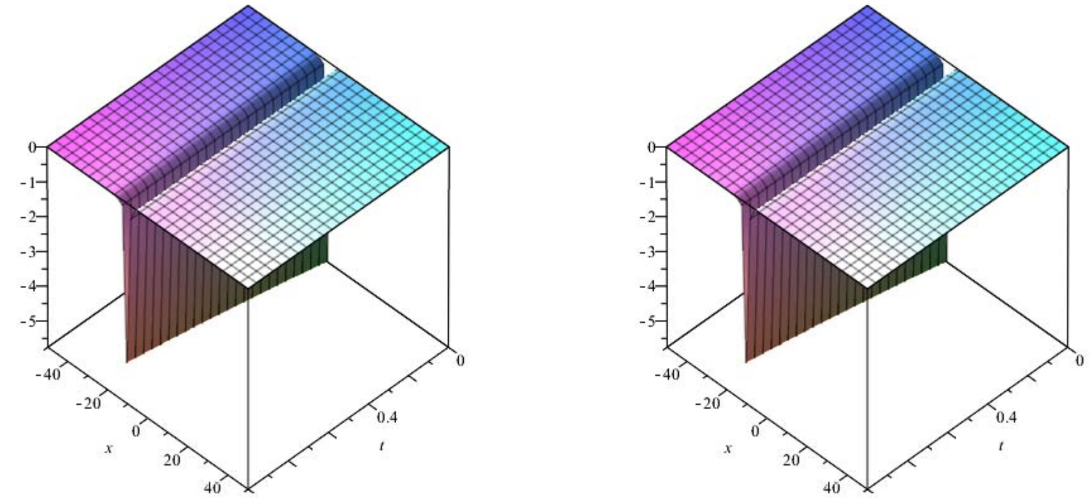
(11)

After five iterations, the graphs of the VIM and exact solutions with different parameters are plotted as shown in Figs. 1 - 4.



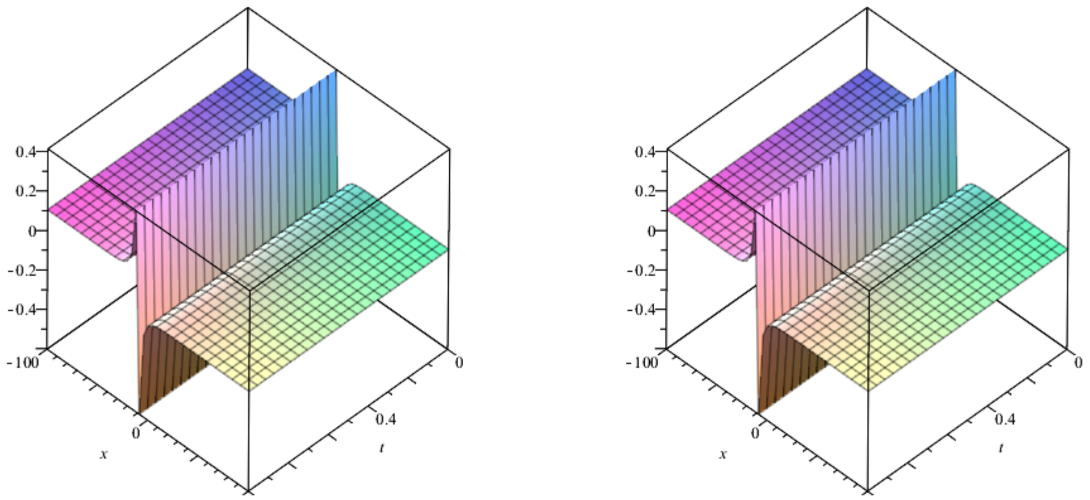
(a) (b)

Figure 1. (a) Exact solution and (b) VIM solution of $u(x,t)$ with $-50\le x\le 50$, $0\le t\le 1$, $k=0.1$, and $x_0=10$.



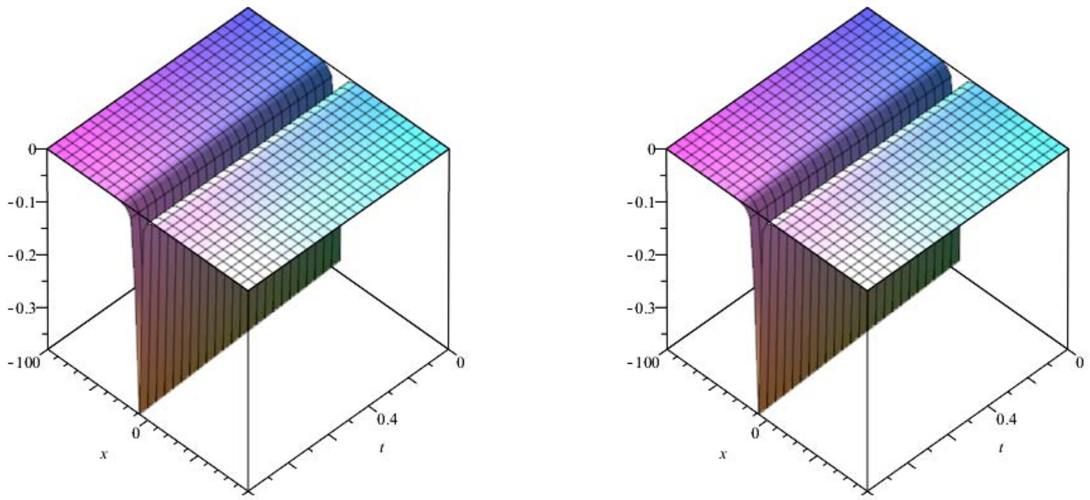
(a) **(b)**

Figure 2. (a) Exact solution and (b) VIM solution of $v(x,t)$ **with** $-50 \le x \le 50$**,** $0 \le t \le 1, k = 0.1$**, and** $x_0 = 10$**.**



(a) **(b)**

Figure 3. (a) Exact solution and (b) VIM solution of $u(x,t)$ **with** $-100 \le x \le 100$**,** $0 \le t \le 1, k = 0.1$**, and** $x_0 = 10$**.**



(a) **(b)**

Figure 4. (a) Exact solution and (b) VIM solution of $v(x,t)$ **with** $-100 \le x \le 100$**,** $0 \le t \le 1, k = 0.1$**, and** $x_0 = 10$**.**

Example 3.2: Consider the WBK (1) with $\alpha = 0.5$ and $\beta = 1$ then the equation becomes:

$$u_t + uu_x + v_x + u_{xx} = 0$$
$$v_t + uv_x + vu_x + 0.5u_{xxx} - v_{xx} = 0 \tag{12}$$

with the initial conditions:

$$\begin{aligned} u(x,t) &= \lambda - 2\sqrt{1.5}k \coth[k(x + x_\circ)] \\ v(x,t) &= -2\sqrt{3}k^2 \operatorname{csc}h^2[k(x + x_\circ)] \end{aligned} \tag{13}$$

The iteration formula as becomes:

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - \int_0^t \left[u_n(x,\xi)_\xi + u_n(x,\xi)u_n(x,\xi)_x + v_n(x,\xi)_x + u_n(x,\xi)_{xx}\right]d\xi \\ V_{n+1}(x,t) &= v_n(x,t) - \int_0^t \begin{bmatrix} u_n(x,\xi)_\xi + u_n(x,\xi)v_n(x,\xi)_x + v_n(x,\xi)u_n(x,\xi)_x + \\ 0.5u_n(x,\xi)_{xxx} - v(x,\xi)_{xx} \end{bmatrix} d\xi \end{aligned} \tag{14}$$

Using the above iteration, the following approximation can be obtained.

$$\begin{aligned} u_0(x,t) &= \lambda - 2.449489742k \coth(k(x + x_0)) \\ v_0(x,t) &= \lambda - 2.724744871k^2 \operatorname{csc}h(k(x + x_0))^2 \end{aligned} \tag{15}$$

$$\begin{aligned} u_1(x,t) &= \lambda - 2.449489742\, \mathrm{k}\coth(k(x + x_0)) + 2.449489742k^2 t\lambda \\ &- 2.44948974k^2 t\lambda \coth(k(x + x_0))^2 - 10.89897948k^3 t \coth(k(x + x_0)) \\ &+ 10.89897948k^3 t \coth(k(x + x_0))^3 - 5.449489742k^3 \operatorname{csc}h(k(x + x_0))^2 \coth(k(x + x_0))t \end{aligned}$$

$$\begin{aligned} v_1(x,t) &= -2.724744871k^2 \operatorname{csc}h(k(x + x_0))^2 - 5.449489742k^3 \operatorname{csc}h(k(x + x_0))^2 \\ &\coth(k(x + x_0))t\lambda + 20,02270383k^4 \operatorname{csc}h(k(x + x_0))^2 \coth(k(x + x_0))^2 t \\ &- 6.674234611k^4 \operatorname{csc}h(k(x + x_0))^2 t - 2.449489742k^4 t \\ &+ 9.707958968k^4 t \coth(k(x + x_0))^2 - 7.348469226k^4 t \coth(k(x + x_0))^4 \\ &+ 4.898979484k^3 t \coth(k(x + x_0)) - 4.898979484k^3 t \coth(k(x + x_0))^3 \end{aligned} \tag{16}$$

After five iterations, the graphs of the VIM and exact solutions with different parameters are plotted as shown in Figs. 5 to 8.

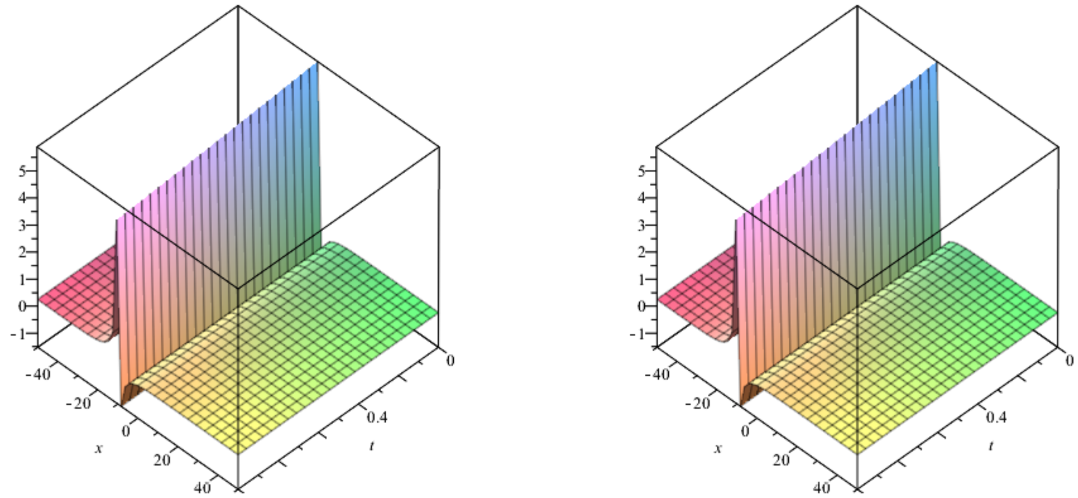


Figure 5. (a) Exact solution and (b) VIM solution of $u(x,t)$ with $-50 \le x \le 50$, $0 \le t \le 1, k = 0.1$, and $x_0 = 10$.

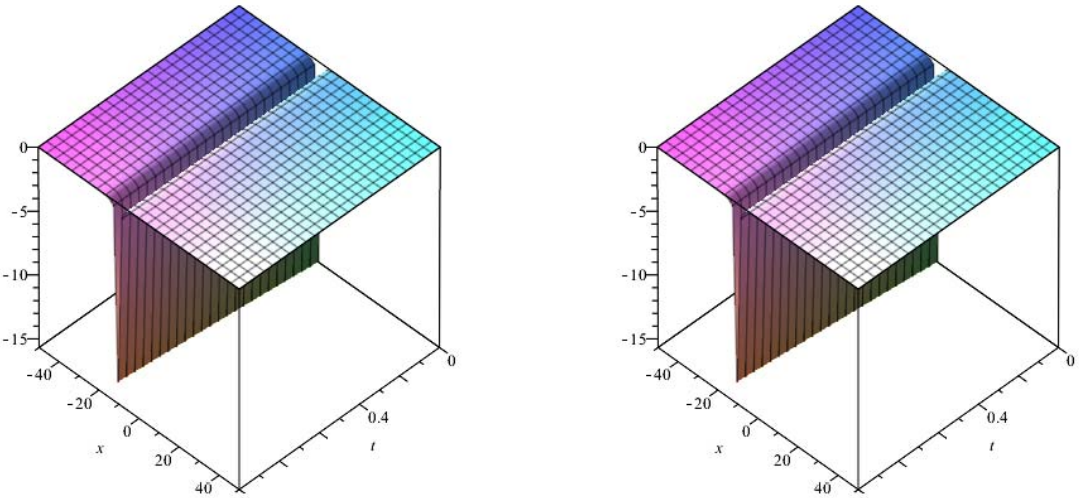


Figure 6. (a) Exact solution and (b) VIM solution of $v(x,t)$ with $-50 \le x \le 50$, $0 \le t \le 1, k = 0.1$, and $x_0 = 10$.

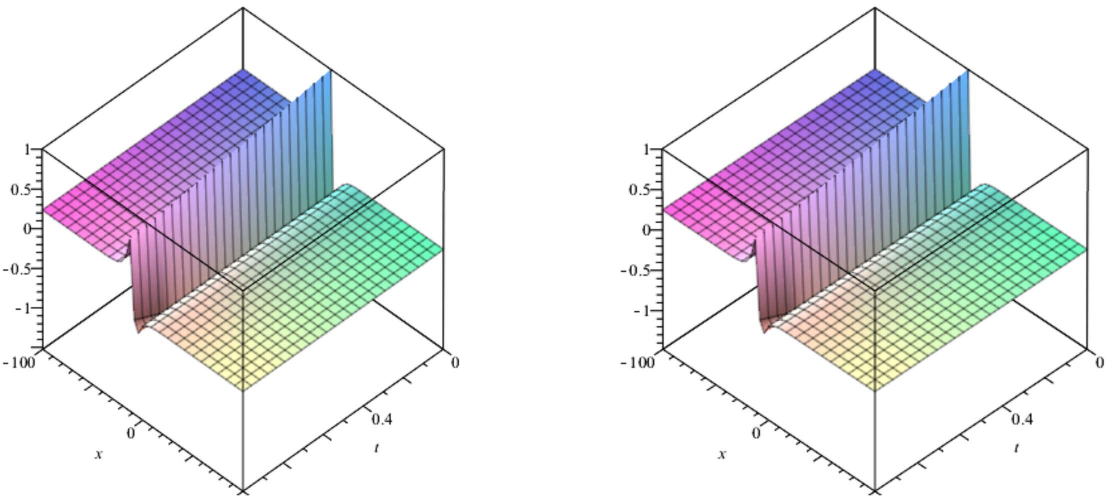


Figure 7. (a) Exact solution and (b) VIM solution of $u(x,t)$ with $-100 \le x \le 100$, $0 \le t \le 1, k = 0.1$, and $x_0 = 10$.

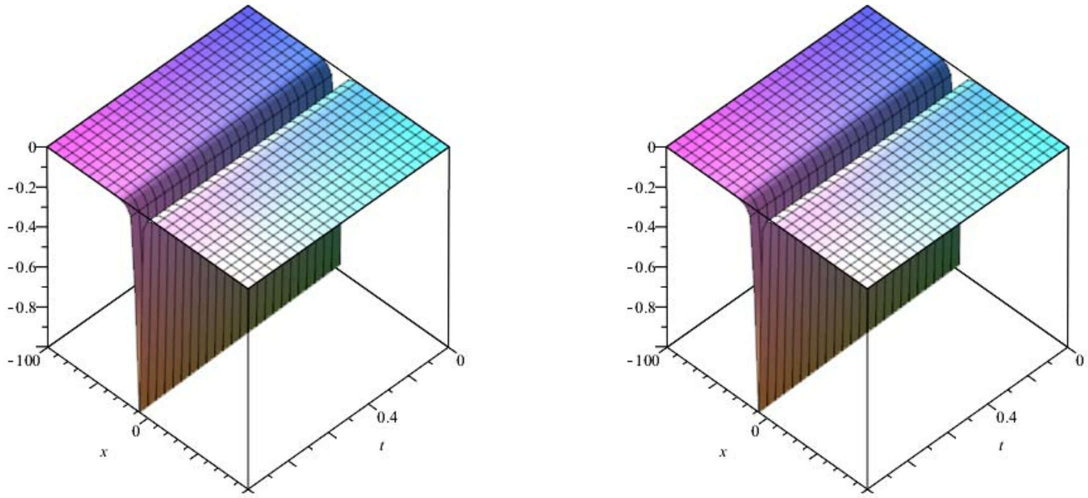


Figure 8. (a) Exact solution and (b) VIM solution of $v(x,t)$ with $-100 \le x \le 100$, $0 \le t \le 1$, $k = 0.1$, and $x_0 = 10$.

CONCLUSION

In this paper, the variational iteration method has been used to find the numerical solutions of the Whitham-Broer-Kamp (WBK) equations in shallow water. The solution obtained was in good agreement with the exact solutions as shown in the graphs plotted. This method does not require any linearization and discretization. The method also requires less computational time as it converges rapidly. It was also observed that the VIM is powerful tool for solving WBK equations with wide applications in sciences and engineering.

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