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NEW DERIVATIVE FREE ITERATIVE METHOD'S FOR SOLVING NONLINEAR EQUATIONS USING STEFFENSEN'S METHOD

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Abstract.*In this paper, we introduce the comparative study of derivative free new two step iterative method for finding the zeros of the nonlinear equation $f(x) = 0$ without the evaluation of the derivatives . It is established that the new method has convergence order three. The efficiency index of new method is equal to 1.442. The Convergence and error analysis are given. Numerical comparisons are made with other existing methods to show the performance of the presented methods.*

Keywords: *Nonlinear equations, Iterative methods, Two step, Derivative free method, Steffensen's method.*

1. INTRODUCTION

It is well known a wide class of linear and nonlinear problems which arise in different branches of mathematical such as physical, biomedical, regional, optimization, ecology, economics and engineering sciences can be formulated in terms of nonlinear equations . Iterative methods for finding the approximate solution of the nonlinear equation $f(x) = 0$ are being developed using several different techniques including Taylor series, quadrature formulas,homotopy and decomposition techniques, see [3, 5-7, 9, 10,13-23] and the references therein . Inspired and motivated by the ongoing research activities in this area, we suggest and analyze a new iterative method for solving nonlinear equations.

In recent years many researchers have developed several iterative methods for solving nonlinear equations. In this paper we are going to developed efficient methods to find approximations of the root α of

$$f(x) = 0 \tag{1}$$

The most famous of these methods is the classical Newton's method (NM) [11] .

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Therefore the Newton's method was modified by Steffensen's method who replaced the first derivative $f'(x)$ in Newton's method by forward differences approximation.[11]

$$f'(x) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)} = P_0(x_n) \tag{2}$$

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And obtained the famous Steffensen's method (S M) [11]

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n))-f(x_n)} \quad (3)$$

Newton and Steffensen's methods are of second order converges, both require two functional evaluations per step , but in contrast to Newton' s method, Steffensen's method is free from any derivative of the function because sometimes the applications of the iteration method which depends upon derivatives are restricted in engineering .

Recently, Cordero et al.[11,12] proposed a derivative free iterative method by replacing the forward difference approximation. We consider the definition of efficiency index as $P^{\frac{1}{n}}$ where P is the order of method and n is the total number of function and derivative evaluations at each step of the iteration. Obviously, if the value of the index is large, then the method more efficient [8], The efficiency index of Newton' s method is 1.414 and the efficiency index of Steffensen' s method is 1.414 .

We use Predictor – corrector methods, we shall now discuss the application of the explicit and implicit multistep methods, for the solution of the initial value problems. We use explicit (predictor) method for predicting a value and then use the implicit (corrector) method iteratively until the convergence is obtained[1] .

2. ITERATIVE METHOD

Numerical analysis is the area of mathematics and computer sciences, which arise in various fields of pure and applied sciences can be formulated in terms of nonlinear equation of the type.

Recently, Najmuddin Ahmad and Vimal Pratap Singh [2] have obtained some new two step iterative methods for solving non linear equations using Steffensen's method (NTS-2)

NTS-2 : For a given x_0 , compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n))-f(x_n)}$$

$$x_{n+1} = x_n - \frac{2\ f(x_n)}{[f'(y_n)+f'(x_n)]}$$

$$n = 0, 1, 2, 3, \ldots$$

NTS - 2 has second order convergence.

A significant part in developing our new iterative methods free from first derivative with respect to y. To be more precise, we now approximate $f'(y_n)$, to reduce the number of evaluations per iteration by a combination . Toward this end, an estimation of the function $P_1(t)$ is taken into consideration as follows:

$$P_1(t) = a + b\,(t - x_n) + c\,(t - x_n)^2 ,$$

$$P_1'(t) = b + 2c\,(t - x_n) ,$$

By substituting in the known values

$$P_1(y_n) = f(y_n) = a + b\,(y_n - x_n) + c\,(y_n - x_n)^2 ,$$

$$P_1'(y_n) = f'(y_n) = b + 2c\,(y_n - x_n) ,$$

$$P_1(x_n) = f(x_n) = a , \quad P_1'(x_n) = f'(x_n) = b ,$$

we could easily obtain the unknown parameters . Thus we have

$$f'(y_n) = 2\left(\frac{f(y_n) - f(x_n)}{y_n - x_n}\right) - f'(x_n) = P_1(x_n , y_n) , \tag{4}$$

Now using Equations (1) – (4) to suggest the following new derivative free iterative methods for solving nonlinear equation , It is established that the following new methods have convergence order three , which will denote by Ahmad Singh Methods (ASM) Then Theorem 2.1 can be written in the following form.

Theorem 2.1 :For a given x_0 , compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)\,(y_n - x_n)}{f(y_n) - f(x_n)}$$

n= 0, 1, 2, 3, …………..

It is known as ASM.

3. CONVERGENCE ANALYSIS

Let us now discuss the convergence analysis of the above method ASM.

Theorem 3.1:let $\alpha \epsilon I$ be a simple zero of sufficiently differential function $f : I \subseteq R \rightarrow R$ for an open interval I, if x_0 is sufficiently close to α then the two step iterative method defined by theorem 2.1 third order convergence .

*Proof:*Consider to

$$y_n = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}$$

$$x_{n+1} = x_n - \frac{f(x_n)\,(y_n - x_n)}{f(y_n) - f(x_n)}$$

Let α be a simple zero of f. Than by expanding $f(x_n)$about α we have

$$f(x_n) = e_n c_1 + e_n^2 c_2 + e_n^3 c_3 + O(e_n^4) \tag{5}$$

where $c_k = \frac{1}{k!} f^{(k)}(\alpha)$ k=1, 2, 3, …

and $e_n = x_n - \alpha$

from (5) , we have

$$[f(x_n)]^2 = c_1^2 e_n^2 + 2\, c_1 c_2 e_n^3 + c_2^2 e_n^4 + \ldots\ldots. \quad (6)$$

$$f\big(x_n + f(x_n)\big) = c_1(1+c_1)e_n + (3c_1c_2 + c_1^2c_2 + 2c_2^2)e_n^2 + \ldots \quad (7)$$

$$f\big(x_n + f(x_n)\big) - f(x_n) = c_1^2 e_n + (3c_1c_2 + c_1^2c_2 + 2c_2^2)e_n^2 + \ldots \quad (8)$$

Now by substituting (6) and (8) in y_n, we have

$$y_n = \alpha + \left(\frac{c_2}{c_1} + c_2\right)e_n^2 + \left(2\frac{c_3}{c_1} + c_3c_1 - c_2^2 + 3c_3 - 2\frac{c_2^2}{c_1^2}\right)e_n^3 + \ldots \quad (9)$$

By using Taylor's theorem, we have

$$f(y_n) = c_2(1+c_1)e_n^2 + \left(2c_3^3c_1 + c_3c_1^3 - c_2^2c_1^2 + 3c_3c_1^2 - 2c_2^2 - 2c_2^2c_1 \right)\frac{e_n^3}{c_1} + \ldots \quad (10)$$

$$y_n - x_n = -e_n + \left(\frac{c_2}{c_1} + c_2\right)e_n^2 + \left(2\frac{c_3}{c_1} + c_3c_1 - c_2^2 + 3c_3 - 2\frac{c_2^2}{c_1^2}\right)e_n^3 + \ldots \quad (11)$$

$$f(y_n) - f(x_n) = -c_1e_n + c_1c_2e_n^2 + \left(2c_3^3c_1 + c_3c_1^3 - c_2^2c_1^2 + 3c_3c_1^2 - 2c_2^2 - 2c_2^2c_1 - \right.$$
$$\left. - c_3c_1\right)\frac{e_n^3}{c_1} + \ldots \quad (12)$$

$$f(x_n)(y_n - x_n) = -c_1e_n^2 + c_1c_2e_n^3 + \left(c_3 - c_2^2 - \frac{c_2^2}{c_1} + c_3c_1^2 - c_1c_2^2 + 3c_3c_1\right)e_n^4 + \ldots \quad (13)$$

$$\frac{f(x_n)(y_n - x_n)}{f(y_n) - f(x_n)} = e_n - c_2^2e_n^3 - O(e_n^4) \quad (14)$$

$$x_{n+1} = \alpha + c_2^2e_n^3 + O(e_n^4) \quad (15)$$

$$e_{n+1} = c_2^2e_n^3 + O(e_n^4)$$

This shows that Theorem 2.1 is third order convergence.

4. NUMERICAL EXAMPLES

For comparisons, we have used Newton's method (NM) [11]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

Steffensen's method (SM) [11]

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)},$$

And Dehghan method (DM) (4)

$$y_n = x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n))-f(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)[f(x_n)+f(y_n)]}{f(x_n+f(x_n))-f(x_n)}$$

$f_1(x) = x\tan(x) + 1$
$f_2(x) = \sin(x) - 1 + x$
$f_3(x) = \cos(x) - \sqrt{x} + 1$
$f_4(x) = x^3 - 6x + 4$
$f_5(x) = \sin(x) - 1 - x^3$
$f_6(x) = 4x - e^x$
$f_7(x) = e^x - 1.5 - tan^{-1}(x)$
$f_8(x) = x - \cos(x)$
$f_9(x) = xe^x - 2$
$f_{10}(x) = xlog_{10}x - 1.2$

As for the convergence criteria, it was required that the distance of two consecutive approximations δ and also displayed is the number of iterations to approximate the zero (IT), the approximate zero x_n and the value $f(x_n)$.

Table 1.Numerical examples and comparison.

Method	**IT**	x_n	$f(x_n)$	δ
$f_1, x_0 = 2.5$			-7.16049482103465e-005	
NM	3	2.77906483703391		4.63531457597366e-005
SM	6	2.77906483703391		4.63531457597366e-005
DM	4	2.77906483703391		0.000135498817119828
ASM	3	2.77906483703391		7.19834991613766e-009
$f_2, x_0 = 0.8$			-1.11022302462516e-016	
NM	4	0.510973429388569		3.78060912575862e-008
SM	5	0.510973429388569		4.42400560629608e-010
DM	3	0.510973429388569		8.57036115964327e-007
ASM	3	0.510973429388569		1.47768097757961e-009
$f_3, x_0 = 1$			2.44249065417534e-015	
NM	4	1.39058983057829		1.39888101102769e-014
SM	3	1.39058983057829		1.83251690488717e-008
DM	2	1.39058983057829		2.3604739289862e-005
ASM	2	1.39058983057829		3.2408545299844e-005

Method	**IT**	x_n	$f(x_n)$	δ
$f_4, x_0 = 1$			8.88178419700125e-016	
NM	4	0.732050807568677		1.16352300016942e-008
SM	6	0.732050807568677		5.42884504017849e-009
DM	5	0.732050807568677		5.66990898676067e-013
ASM	4	0.732050807568677		1.19904086659517e-014
$f_5, x_0 = -1$			-1.99840144432528e-014	
NM	4	-1.24905214850119		0.000227912085001725
SM	6	-1.24905214850119		7.16086545615724e-009
DM	4	-1.24905214850119		4.26220170268721e-010
ASM	4	-1.24905214850119		1.25599530775844e-011
$f_6, x_0 = 1.8$			-1.77635683940025e-015	
NM	6	2.15329236411035		6.50146603220492e-013
SM	6	2.15329236411035		1.48246099840321e-008
DM	4	2.15329236411035		8.48793630048306e-007
ASM	4	2.15329236411035		1.82088919942203e-007
$f_7, x_0 = 1$			0	
NM	5	0.767653266201279		2.27928786955545e-012
SM	6	0.767653266201279		1.24644738974666e-012
DM	4	0.767653266201279		3.84956955201687e-010
ASM	4	0.767653266201279		9.99200722162641e-016
$f_8, x_0 = 0.5$			6.66133814775094e-016	
NM	4	0.739085133215161		1.70123026776992e-010
SM	5	0.739085133215161		7.13984427136438e-013
DM	3	0.739085133215161		1.69392880700059e-006
ASM	3	0.739085133215161		3.30882599097748e-009
$f_9, x_0 = 1$			2.22044604925031e-015	
NM	4	0.852605502013726		2.43551959711041e-008
SM	6	0.852605502013726		3.5176106472079e-010
DM	4	0.852605502013726		1.5360264060682e-008
ASM	3	0.852605502013726		9.99200722162641e-016

Method	IT	x_n	$f(x_n)$	δ
$f_{10}, x_0 = 2$ NM SM DM ASM	 4 5 3 3	 2.74064609597369 2.74064609597369 2.74064609597369 2.74064609597369	-3.10862446895044e-015	1.70161289503313e-008 1.16928688953521e-012 7.48084598978238e-006 3.04622895797024e-008

5. CONCLUSION

In this paper, we have suggested and analyzed new derivative free two step iterative method for finding the zeros of nonlinear equations . This method based on a steffensen's method and using predictor - corrector technique. The error equations are given theoretically to show that the proposed technique third order convergence. The new method attain efficiency index of 1.442. Which makes it competitive. In addition, numerical tests show that the new method is comparable with the well known existing methods and gives better result.

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