ORIGINAL PAPER

STUDY OF DIELECTRIC RELAXATION PROCESS EQUATION USING MODIFIED REIMANN-LIOUVILLE DERIVATIVE

JAMSHAD AHMAD[1], MUHAMMAD NAEEM[2]

Manuscript received: 26.05.2016; Accepted paper: 12.09.2016;
Published online: 30.12.2016.

Abstract. *In this study, a reliable Homotopy Analysis Method (HAM) is applied to obtain analytical solutions of dielectric relaxation processes equation of fractional order. The, fractional derivatives are based on Jumarie's fractional derivative sense. The obtained solutions which are valid at low temperatures are applied to the dielectric relaxation processes. The graphical representation of the obtained results is courageous.*
Keywords: *Dielectric Relaxation Process, Fractional differential equation, Modified Reimann-Liouville Derivative, Homotopy Analysis Method, Ising Model.*

1. INTRODUCTION

In recent years, the analysis of fractional differential equations, which are obtained from the classical differential equations in mathematical physics, engineering, vibration and oscillation by replacing the second order time derivative by a fractional derivative of order α satisfying $0 < \alpha \leq 1$, have been a field of growing interest as evident from literature survey [1-6]. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

G. Jumarie [7, 8] introduced a new definition of modified Riemann-Liouville derivative.This modified definition of Riemann-Liouville derivativein comparison with the classical caputo derivative, is not required to satisfy higher integer-order derivative than α. Secondly α th derivative of a constant is zero. Due to these merits, this new modified definition of fractional derivative were successfully applied in the probability calculus [9], fractional Laplace problem [10].

The solution of a fractional differential equation is much involved. In general, there exists no method that yields an exact solution for a fractional differential equation. Only approximate solutions can be derived using the linearization or perturbation methods. The Homtopy anlysis method is relatively new approach providing an analytical approximation to linear and nonlinear problems, and is particularly valuable as tool for scientists, engineers, and applied mathematicians, because it provide immediate and visible symbolic terms of analytic solutions, as well as a numerical approximate solution to both linear and nonlinear differential equations without linearization or discretization.

In this paper, we appliedthe Homotopy Analysis Method (HAM) to obain the analtical approximate solutions of fractional order relaxation of dielectric materials equation. This fractional relaxation of dielectric materials equation is obtained by replacing the time

[1]Department of Mathematics, Faculty of Sciences, University of Gujrat, Pakistan.
E-mail: jamshadahmadm@gmail.com.
[2]Department of Mathematics, Faculty of Sciences, UET Lahore, Pakistan.

derivative term in the corresponding vibration equation by a fractional derivative of order α with $0 < \alpha \le 1$. The derivatives are understood in the modified Riemann-Liouville sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 1$, the fractional relaxation of dielectric materials equation reduces to the standard vibration equation. By the present method, numerical results can be obtained with using a few iterations.The homotopy analysis method (HAM) contains the auxiliary parameter $\hbar$. which provides us with a simple way to adjust and control the convergence region of solution series for large value of t [11, 12] .Unlike, other numerical methods which give low degree of accuracy for large values of t.Therefore, the homotopy analysis method (HAM) handles linear and inhomogeneous problems without any assumption and restriction [13].

2. MODIFIED RIEMANN-LIOUVILLE DERIVATIVE

Assume $f : R \to R, x \to f(x)$ denote a continuous (but not necessarily differentiable) function and let the partition $h > 0$ in the interval [0, 1]. Through the fractional Riemann Liouville integral

$$ {}_0I_x^{\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-\xi)^{\alpha-1} f(\xi)d\xi \ , \ \alpha > 0, \tag{1}$$

The modified Riemann-Liouville derivative is defined as

$$ {}_0D_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-\xi)^{n-\alpha}(f(\xi)-f(0))\,d\xi, \tag{2}$$

where $x \in [0,1]$, $n-1 \le \alpha < n$ and $n \ge 1$.

Jumarie's modified fractional derivative is given by

$$ \Delta^{\alpha} = (FW-1)^{\alpha} f(x) = \sum_0^{\infty}(-1)^k \binom{\alpha}{k} f[x+(\alpha-k)h], \tag{3}$$

where $FWf(x) = f(x+h)$. The fractional derivative is defined as,

$$ f^{(\alpha)} = \lim_{h\to 0}\frac{\Delta^{\alpha} f(x)}{h^{\alpha}}. \tag{4}$$

The proposed Jumarie's modified Riemann-Liouville derivative given in Eq. (2) is strictly equivalent to Eq. (4). Some properties are given by Fractional Leibniz product law

$$ {}_0D_x^{\alpha}(uv) = u^{(\alpha)}v + uv^{(\alpha)} \tag{5}$$

Fractional Leibniz formulation

$$ {}_0I_x^{\alpha}\,{}_0D_x^{\alpha} f(x) = f(x) - f(0), \ 0 < \alpha \le 1 \tag{6}$$

The integration by parts formula is

$$ {}_aI_b^{\alpha} u^{(\alpha)} v = (uv)/_a^b - {}_aI_b^{\alpha} u v^{(\alpha)} \tag{7}$$

Integration with respect to $(d\xi)^{\alpha}$.

Assume $f(x)$ denote a continuous $R \to R$ function, we use the following quality for the integral with respect to $(d\xi)^{\alpha}$

$$ {}_0I_x^{\alpha} f(x) = \frac{1}{\Gamma\alpha}\int_0^x (x-\xi)^{\alpha-1} f(\xi)\, d\xi = \frac{1}{\Gamma(1+\alpha)}\int_0^x f(\xi)(d\xi)^{\alpha}. \tag{8}$$

3. THE ANALYSIS OF HOMOTOPY ANALYSIS METHOD (HAM)

Consider the following general differential equation

$$ FD[u(x,t)] = 0, \tag{9}$$

where*FD* is a nonlinear differential operator for this problem. According tohomotopy analysis method, the zeroth-order deformation equation is

$$ (1-q)L(U(x,t;q) - u_0(x,t)) = q\hbar H(x,t) FD(U(x,t;q)), \tag{10}$$

where $q \in [0, 1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, $H(x,t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(x,t)$is an initial guess of $u(x,t)$ and $U(x,t;q)$ is an unknown function of the independent variables x,t and q.

Obviously, when $q = 0$ and $q = 1$, it holds

$$ U(x,t;0) = u_0(x,t), U(x,t;1) = u(x,t), \tag{11}$$

respectively. Using the parameter *q*, we expand $U(x,t;q)$ in Taylor series as follows:

$$ U(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\, q^m, \tag{12}$$

where

$$ u_m = \frac{1}{m!}\frac{\partial^m U(t;q)}{\partial^m q}\bigg|_{q=0} $$

The auxiliary linear operator, the initial guess, the auxiliary parameter$\hbar$ and the auxiliary function $H(x,t)$ are selected in away that the Eq. (12) is convergent at$q = 1$, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t), \tag{13}$$

We define

$$\vec{u}_n(x,t) = \{u_0(x,t), u_1(x,t), \ldots, u_n(x,t)\}$$

Differentiating Eq. (10) $m$times with respect to parameter$q$, then setting $q = 0$ and finally dividing them by $m!$, the mth-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar H(x,t) R_m(\vec{u}_{m-1}), \tag{14}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} FD(U(t;q))}{\partial^{m-1} q}\bigg|_{q=0},$$

and

$$\chi_m = \begin{cases} 0 & m \le 1, \\ 1 & m > 1. \end{cases}$$

Finally, we will approximate the HAM solution of Eq. (9) by the following truncated series:

$$\phi_m(t) = \sum_{k=0}^{m-1} u_k(t).$$

4. DIELECTRIC RELAXATION PROCESSES

The relaxation property is generally expressed in terms of time-domain response function $f(t)$ [14]:

$$\tilde{f}(i\,\omega) = \int_0^{\infty} e^{-i\omega} f(t)\,dt \tag{15}$$

$$= \varphi'(\omega) - i\,\varphi''(\omega) \tag{16}$$

Classically, relaxation processes are described in terms of the exponential function;

$$\phi(t) = \exp(-t/\tau),\ t \ge 0, \tag{17}$$

which is generally referred to as Maxwell–Debye relaxation. However, in many systems the dynamical behavior shows conspicuous deviations from the ideal exponential pattern. Experimental results in the time domain are often described in terms of the Kohlrausch-Williams-Watts (KWW) or stretched exponential function [15]

$$\phi(t) = \exp(-t/\tau)^{\beta},\ 0 < \beta < 1, \tag{18}$$

or through asymptotic power-laws

$$\phi(t) = \frac{1}{1+(t/\tau)^{\delta}},\ \delta > 0, \tag{19}$$

Usually,there are three general relaxation laws for the studies of complex systemsstretched exponential (KWW function) [16]

$$f(t) \approx \exp\left(-(t/\tau)^{\alpha}\right),\ 0<\beta<1, t>\tau, \tag{20}$$

exponential-logarithmic function

$$f(t) \approx \exp\left(-B\ln^{\alpha}(t/\tau)^{\alpha}\right), \tag{21}$$

algebraic decay

$$f(t) \approx (t/\tau)^{-\alpha}, \tag{22}$$

where α, τ, and B are the appropriate fitting parameters [17].

By definition, $\chi(\omega)$ is connected to the temporal relaxation function through the following relation

$$\begin{aligned}\chi(\omega) &= \int_0^{\infty} e^{-i\omega t} d(-\phi(t)) \\ &= i\,\omega - \int_0^{\infty} e^{-i\omega t}\Phi(t)\,dt\end{aligned} \tag{23}$$

where $\Phi(t) = \phi(t)/\phi(0)$. Significant amount of experimental data on disordered systems supports the following empirical expressions for dielectric loss spectra, namely, the Cole-Cole equation [18]

$$\chi(\omega) = \frac{\chi_0}{1+(i\,\omega\tau)^{\alpha}},\ 0<\alpha\le 1, \tag{24}$$

the Cole-Davidson equation [19]

$$\chi(\omega) = \frac{\chi_0}{1+(i\,\omega\tau)^{\beta}},\ 0<\beta\le 1, \tag{25}$$

and the Havriliak-Negami equation [20] considered as a general expression for the universal relaxation law.

$$\chi(\omega)=\frac{\chi_0}{\left(1+(i\,\omega\tau)^{\alpha}\right)^{\beta}},\ 0<\alpha\le 1,\ 0<\beta\le 1, \tag{26}$$

Havriliak-Negami equation is a combination of the Cole-Cole and Cole-Davidson equations.

5. THE ISING MODEL AND FRACTIONAL RELAXATION

The decay of the spin-spin time correlation functions in a one-dimensional Ising model with Glauber [21] dynamics was studied by Brey and Parados[22]. They started that the energy of the system in the one-dimensional Ising model for a configuration σ is

$$H(\sigma)=-J\sum_i \sigma_i\sigma_{i+1} \tag{27}$$

with J a positive constant. The state of the system is specified by the spin vector $\sigma=\{\sigma_i\}$, where $\sigma_i=\pm 1$ is the spin at site i. The evolution of the system is described by Markov process with Glauber dynamics. So, the conditional probability $P_{1/1}(\sigma,t/\sigma',t')$ of finding the system in the state σ at time t, given it was in the state σ' at time t' obeys the master equation

$$\frac{\partial P_{1/1}(\sigma,t/\sigma',t')}{\partial t}=\sum_{i=-\infty}^{\infty}\left(\omega_i(R_i\sigma)P1/1(R_i\sigma,t/\sigma',t')-\omega_i(\sigma)P1/1(\sigma,t/\sigma',t')\right) \tag{28}$$

where $R_i\sigma$ is the configuration obtained from σ by flipping the i.th spin and $\omega_i(\sigma)$ is the transition rate for the flip. Following, in the low temperature limit, spin-spin time correlation function was found by Brey and Parados [22] in form of a diffusion type equation

$$\frac{\partial u(x,t)}{\partial t}=(\alpha\gamma-\alpha)f(x,t)+\frac{\alpha\gamma}{2}\frac{\partial^2 u(x,t)}{\partial t^2}. \tag{29}$$

If Eq. (29) is evaluated to fractional differential equation form, the one can be expressed as

$$D_t^{\xi}u(x,t)=(\alpha\gamma-\alpha)f(x,t)+\frac{\alpha\gamma}{2}\frac{\partial^2 u(x,t)}{\partial t^2}. \tag{30}$$

where D_t^{ξ} is the Riemann-Liouville fractional differential operator, and the initial condition for this equation is

$$u(x,0)=e^{-|x|}. \tag{31}$$

6. APPLICATION OF HOMOTOPY ANALYSIS METHOD

According to eq. (10), the zeroth-order deformation can be given by

$$(1-q)L(U(x,t;q)-u_0(x,t)) = q\hbar H(x,t)\left(D_t^{\alpha}U(x,t;q)-(\alpha\gamma-\alpha)u(x,t)-\frac{\alpha\gamma}{2}\frac{\partial^2 U(x,t;q)}{\partial x^2}\right), \tag{32}$$

We can start with an initial approximation $u_0(x,t)=e^{-|x|}$ and we choose the auxiliary linear operator

$$L(U(x,t;q)) = D_{\alpha}^{t}U(x,t;q),$$

with the property $L(C)=0,$ where C is an integral constant. We also choose the auxiliary function to be $H(x,t)=1$.

Hence, the mth-order deformation can be given by

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)] = \hbar H(x,t)R_m(\vec{u}_{m-1}),$$

where

$$R_m(\vec{u}_{m-1}) = D_{\alpha}^{t}(u_{m-1})-(\alpha\gamma-\alpha)u_{m-1}-\frac{\alpha\gamma}{2}\frac{\partial^2 u_{m-1}}{\partial x^2} \tag{33}$$

The solution of the mth-order deformation Eq. (14) for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t)+\hbar L^{-1}[R_m(\vec{u}_{m-1})]. \tag{34}$$

Consequently, we have

$$u_0(x,t) = e^{-|x|},$$

$$u_1(x,t) = \left(\alpha\gamma-\alpha+\frac{\alpha\gamma}{2}\right)\frac{e^{-|x|}t^{\xi}}{\Gamma(1+\xi)},$$

$$u_2(x,t) = \left(\alpha\gamma-\alpha+\frac{\alpha\gamma}{2}\right)^2\frac{e^{-|x|}t^{2\xi}}{\Gamma(1+2\xi)},$$

$$u_3(x,t) = \left(\alpha\gamma-\alpha+\frac{\alpha\gamma}{2}\right)^3\frac{e^{-|x|}t^{3\xi}}{\Gamma(1+3\xi)},$$

$$\vdots$$

Hence, the HAM series solution is

$$u(x,t) = u_0(x,t)+u_1(x,t)+\ldots$$

$$u(x,t) = e^{-|x|}+\left(\alpha\gamma-\alpha+\frac{\alpha\gamma}{2}\right)\frac{e^{-|x|}t^{\xi}}{\Gamma(1+\xi)}+\left(\alpha\gamma-\alpha+\frac{\alpha\gamma}{2}\right)^2\frac{e^{-|x|}t^{2\xi}}{\Gamma(1+2\xi)}+\left(\alpha\gamma-\alpha+\frac{\alpha\gamma}{2}\right)^3\frac{e^{-|x|}t^{3\xi}}{\Gamma(1+3\xi)}+\ldots..$$

$$= e^{-|x|}E_{\xi}\left(\alpha\left(-1+\frac{3\gamma}{2}\right)\right) \tag{35}$$

where $\sum_0^\infty \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} = E_\alpha(t^\alpha)$ is the Mittag-Leffler function in one parameter. We can readily check $u(x,t) = E_\alpha(t^\alpha)$is an exact solution of Eq. (30).

If dipoles are located between$x$and $x + x_0$, then probability density given by

$$u(x) = \frac{1}{x_0} e^{(-x/x_0)}. \tag{36}$$

Thus, integrating the dipole correlation function (35) over the all space we can reach to the time dependent correlation function

$$u(t) = \int_0^\infty \frac{1}{x_0} e^{(-x/x_0)} e^{-|x|} E_\xi\left(\alpha\left(-1+\frac{3\gamma}{2}\right)t^\xi\right)dx.$$
$$= \frac{E_\xi\left(\alpha\left(-1+\frac{3\gamma}{2}\right)t^\xi\right)}{x+x_0} \tag{37}$$

where x_0 is the average value of x and $\frac{1}{2x_0}$ is average number of dipoles per unit length.

If Eq. (37) substituted into Eq. (23),

$$\chi(\omega) = 1 - i\omega \int_0^\infty \frac{1}{x_0} e^{-i\omega t} e^{-|x|} \frac{E_\xi\left(\alpha\left(-1+\frac{3\gamma}{2}\right)t^\xi\right)}{x+x_0} dt, \tag{38}$$

at frequency zone, empiric Cole-Cole type equation is obtained simply as:

$$\chi(\omega) = \frac{x_0}{1+(i\omega\tau)^\xi}, \; 0 < \xi < 1, \tag{39}$$

where we take $\tau = \left(\alpha\left(-1+\frac{3\gamma}{2}\right)\right)^{-\xi}$, $\chi_0 = 1 + \lambda(i\omega\tau)^\xi$, and $\lambda = 1 - \frac{1}{x+x_0}$.

Moreover, for sufficiently small times Mittag-Leffler function exhibits the same behavior with a stretched exponential [23].

$$u(t) \approx 1 - \frac{(t/\tau)^\xi}{\Gamma(1+\xi)} + \ldots$$
$$\approx \exp\left(-\frac{(t/\tau)^\xi}{\Gamma(1+\xi)}\right), \; 0 \le t << 1, \tag{40}$$

which is KWW (Kolraush-William-Watts) function. Using the asymptotic expansions it can be written

$$u(t) \approx \frac{\Gamma(\xi)\sin(\xi\pi)}{\pi}(t/\tau)^{-\xi}, \; t \to \infty \tag{41}$$

This is the same form with empirical algebraic decay function (22). When the equation (29) is solved by generation function method at appropriate boundary condition, which was done by [22],

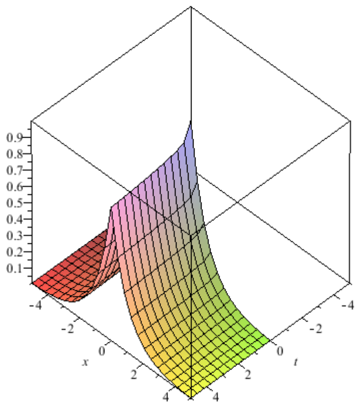
$$\chi(\omega) = \alpha \frac{x_0}{1+\eta^2} \frac{1}{\left((i\,\omega+\alpha)^2 - \alpha^2\gamma^2\right)^{1/2}} \tag{42}$$

is obtained. This expression, at low temperature, converts to Cole-Davidson distribution:

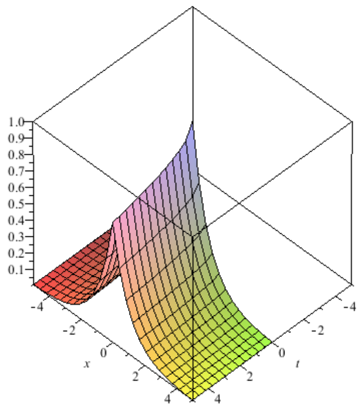
$$\chi(\omega) = \frac{1}{\left(1+i\,\omega\tau_{CD}\right)^{\beta_{CD}}} \tag{43}$$

where τ_{CD} is constant and $\beta_{CD} = \frac{1}{2}$.

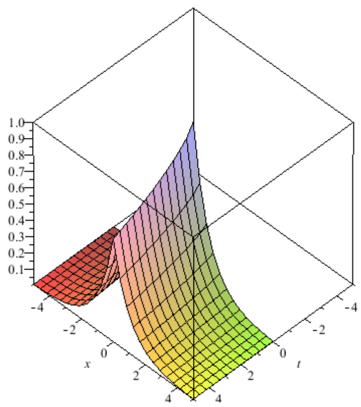
Figures shows the solutions of equation (30) for different values of α.



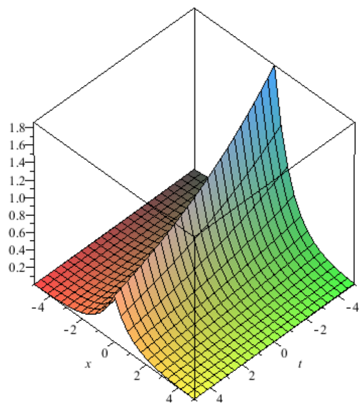
(a) α=.25



(b) α=.50



(c) α=.75



(d) α= 1

7. CONCLUSION

In this study, we applied the Homotopy Analysis Method (HAM) successfully for solving fractional diffusion equation obtained from an evolution of Ising Model. A flexible α parameter, which is especially used in the forming of the differential equations within the fractional order modeling, exhibits that the space of physical processes has a fractional form, and irregularity (or chaos) in the nature compels us to use the fractional theory. From the results, it is concluded that the method proved fullycompatible with the complexity of the problem. A rational level of accuracy reveals the complete reliability and efficiency ofthe algorithm.

REFERENCES

[1] Momani, S.,*Mathematics and Computers in Simulation*, **70**(2), 110, 2005.
[2] Momani, S., Ibrahim, R., *International Journal of Pure and Applied Mathematics*, **37**(1), 119, 2007.
[3] Momani, S., *Applied Mathematics and Computation*, **170**(2), 1126, 2005.
[4] Das, S., *International Journal of Nonlinear Science and Numerical Simulation*, **9**, 361, 2008.
[5] Momani, S., Odibat, Z., Alawneh, A., *Numerical Methods for Partial Differential Equations*, **24**(1), 262, 2008.
[6] Odibat, Z., Momani, S., Alawneh, A., *Journal of Physics: Conference Series*, **96**, 012066, 2008.
[7] Jumarie, G., *International Journal of Systems Science*, **6**, 1113, 1993.
[8] Jumarie, G., *Mathematical and Computer Modelling*, **44**, 231, 2006.
[9] Jumarie, G., *Applied Mathematics Letters*, **22**, 378, 2009.
[10] Jumarie, G., *Applied Mathematics Letters*, **22**, 1659, 2009.
[11] Liao, S.J., *Beyond perturbation: Introduction to the homotopy analysis method*, Chapman and Hall/CRC Press, Boca Raton, 2003.
[12] Liao, S.J., *Journal of Fluid Mechanics*, **488**, 189, 2003.
[13] Liao, S.J., *Communications in Nonlinear Science and Numerical Simulation*, **14**, 983, 2009.
[14] Uchaikin, V.V., *International journal of Theoretical Physics*, **42**(1),121, 2003.
[15] Metzler, R., Klafter, J.,*Journal of Non-Crystalline Solids*, **305**, 81, 2002.
[16] Williams, G., Watts, D.C., Dev, S.B., North, A.M., *Transactions of the Faraday Society*, **67**, 1323, 1971.
[17] Schneider, W.R., Wyss, W., *Journal of Mathematical Physics*, **30**, 134, 1989.
[18] Cole, K.S., Cole, R.N., *Journal of Chemical Physics*, **9**, 341, 1941.
[19] Davidson, D.W., Cole, R.H., *The Journal of Chemical Physics*, **19**, 1484, 1951.
[20] Havriliak, S., Negami, S., *Journal Polymer Science*, **14**(B), 99, 1966.
[21] Glauber, R.G., *Journal of Mathematical Physics*, **4**, 294, 1963.
[22] Brey, J.J., Prados, A., *Physical Review E*, **53**(1), 458, 1996.
[23] Mittag-Leffler, G.M., *Rendiconti: Reale Accademia dei Lincei (Ser. V)*, **13**, 3, 1904.