ORIGINAL PAPER

THE METHOD OF ELEMENTARY OPERATIONS IN BLOCK MATRICES FOR THE DETERMINATION OF THE ANNIHILATOR POLYNOMIALS OF SOME MATRICES

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Abstract. *An efficient method for the characterization of some important classes of matrices, using the rank, is the method of elementary operations in the partitioned matrices. We will use this method for idempotent matrices, involutory matrices, tripotent matrices and other classes of matrices with annihilator polynomials of the form* $f(x) = x^k - x^p$.

Keywords: *Tripotent matrices, elementary block matrices, elementary operations, rank formula for partitioned matrices.*
MSC 2010*: 15A03, 15A18, 15A24.*

1. CHARACTERIZATION OF TRIPOTENT MATRICES

We recall that a matrix $A \in M_n(\mathbf{C})$ is called tripotent if the following relation holds: $A^3 = A$.

Theorem 1.1. *For every matrix* $A \in M_n(\mathbf{C})$ *the following statements are equivalent:*

a) $A^3 = A$;

b) $rank\, A + rank\,(A^2 - I_n) = n$;

c) $rank\,(A - I_n) + rank\,(A^2 + A) = n$;

d) $rank\,(A + I_n) + rank\,(A^2 - A) = n$;

e) $rank\, A + rank\,(I_n - A) + rank\,(I_n + A) = 2n$;

f) $rank\, A = rank\,(A - A^2) + rank\,(A + A^2)$.

Proof. We prove that every assertion is equivalent with a).

If $A^3 = A$ then the eigenvalues of the matrix A satisfy: $\lambda^3 = \lambda$, thus $\lambda \in \{-1, 0, 1\}$.

From the relation $A^3 = A$ it follows that the Jordan canonical form J_A satisfies $J_A^3 = J_A$ and J_A is a diagonal matrix. Thus

$$J_A = \left[\begin{array}{c|c|c} I_k & 0 & 0 \\ \hline 0 & -I_p & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

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where k is the number of eigenvalues equal to 1, p is the number of eigenvalue equal to -1 and $q = n - k - p$ is the number of eigenvalue equal to 0.

In the assertions b), c), d), e, f) we can replace A with J_A, $rank\, A = rank\, J_A$ and we have:

$$rank\, A = k + p\,,\ rank(A^2 - I_n) = q\,,\ rank(A - I_n) = p + q\,,$$
$$rank(A^2 + A) = k\,,\ rank(A + I_n) = k + q\,,\ rank(A^2 - A) = p$$

and all the relations b), c), d), e), f) are satisfied.

For the converse implications: b), c), d), e), f) $\to$ a) we start with block matrix M corresponding to every relation in which we make elementary operations (which preserve the rank), therefore we obtain a matrix N which contains the block $A^3 - A$.

We denote by Q_i the matrices of the elementary operations on columns and by P_i the matrices of elementary operations on rows.

b) $\to$ a) We have

$$M = \begin{bmatrix} A & 0 \\ 0 & A^2 - I_n \end{bmatrix} \xrightarrow{Q_1} \begin{bmatrix} A & A^2 \\ 0 & A^2 - I_n \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} A & A^2 \\ -A & -I_n \end{bmatrix}$$

$$\xrightarrow{Q_2} \begin{bmatrix} A - A^3 & A^2 \\ 0 & -I_n \end{bmatrix} \xrightarrow{P_2} \begin{bmatrix} A - A^3 & 0 \\ 0 & -I_n \end{bmatrix} = N$$

and it follows

$$rank\, M = rank\, A + rank(A^2 - I_n) = rank\, N$$
$$= rank(A - A^3) + rank(-I_n) = rank(A - A^3) + n\,.$$

From b) it follows $rank(A + A^3) = 0$, hence $A^3 = A$. The matrices of elementary operations Q_1, Q_2, P_1, P_2 are

$$Q_1 = \begin{bmatrix} I_n & A \\ 0 & I_n \end{bmatrix},\ P_1 = \begin{bmatrix} I_n & 0 \\ -I_n & I_n \end{bmatrix},\ Q_2 = \begin{bmatrix} I_n & -A \\ 0 & I_n \end{bmatrix},\ P_2 = \begin{bmatrix} I_n & A^2 \\ 0 & I_n \end{bmatrix}$$

so: $N = P_2 \cdot P_1 \cdot M \cdot Q_1 \cdot Q_2$.

c) $\to$ a)

$$M = \begin{bmatrix} I_n - A & 0 \\ 0 & A^2 + A \end{bmatrix} \xrightarrow{Q_1} \begin{bmatrix} I_n - A & -A^2 - A + 2I_n \\ 0 & A^2 + A \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} I_n - A & 2I_n \\ 0 & A^2 + A \end{bmatrix}$$

$$\xrightarrow{P_2} \begin{bmatrix} I_n - A & 2I_n \\ \frac{1}{2}(A^3 - A) & 0 \end{bmatrix} \xrightarrow{Q_2} \begin{bmatrix} 0 & 2I_n \\ \frac{1}{2}(A^3 - A) & 0 \end{bmatrix} = N\,.$$

We have:

$$\begin{aligned} rank\, M &= rank(I_n - A) + rank(A^2 + A) = rank\, N \\ &= rank(2I_n) + rank\,\frac{1}{2}(A^3 - A) = n + rank(A^3 - A) \end{aligned}$$

From c) it follows $rank(A^3 - A) = 0$, hence $A^3 = A$. The matrices of elementary operations are:

$$Q_1 = \begin{bmatrix} I_n & A - 2I_n \\ 0 & I_n \end{bmatrix},\ P_1 = \begin{bmatrix} I_n & I_n \\ 0 & I_n \end{bmatrix},\ Q_2 = \begin{bmatrix} I_n & \frac{1}{2}(A - I_n) \\ 0 & I_n \end{bmatrix},\ P_2 = \begin{bmatrix} I_n & \frac{1}{2}(A^2 + A) \\ 0 & I_n \end{bmatrix}$$

so: $N = P_2 \cdot P_1 \cdot M \cdot Q_1 \cdot Q_2$.

d) $\to$ a)

$$M = \begin{bmatrix} I_n + A & 0 \\ 0 & A^2 - A \end{bmatrix} \xrightarrow{Q_1} \begin{bmatrix} I_n + A & A^2 - A - 2I_n \\ 0 & A^2 - A \end{bmatrix} \xrightarrow{P_1} \begin{bmatrix} I_n + A & -2I_n \\ 0 & A^2 - A \end{bmatrix}$$

$$\xrightarrow{Q_2} \begin{bmatrix} 0 & -2I_n \\ \frac{1}{2}(A^3 - A) & A^2 - A \end{bmatrix} \xrightarrow{P_2} \begin{bmatrix} 0 & -2I_n \\ \frac{1}{2}(A^3 - A) & 0 \end{bmatrix} = N\ .$$

We have

$$\begin{aligned} rank\, M &= rank(I_n + A) + rank(A^2 - A) \\ &= rank\, N = rank\,\frac{1}{2}(A^3 - A) + rank(-2I_n) = rank(A^3 - A) + n \end{aligned}$$

and from d) it follows $rank(A^3 - A) = 0$, hence $A^3 = A$.

e) $\to$ a) We will make elementary operations in a matrix with 9 blocks

$$M = \begin{bmatrix} A & 0 & 0 \\ 0 & I_n - A & 0 \\ 0 & 0 & I_n + A \end{bmatrix} \xrightarrow{Q_1} \begin{bmatrix} A & A & 0 \\ 0 & I_n - A & 0 \\ 0 & 0 & I_n + A \end{bmatrix}$$

$$\xrightarrow{P_1} \begin{bmatrix} A & A & 0 \\ A & I_n & 0 \\ 0 & 0 & I_n - A \end{bmatrix} \xrightarrow{Q_2} \begin{bmatrix} A - A^2 & A & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n + A \end{bmatrix}$$

$$\xrightarrow{P_2} \begin{bmatrix} A - A^2 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n + A \end{bmatrix} \xrightarrow{P_3} \begin{bmatrix} A - A^2 & 0 & A + A^2 \\ 0 & I_n & 0 \\ 0 & 0 & I_n + A \end{bmatrix}$$

$$\xrightarrow{Q_3}\begin{bmatrix} A-A^2 & 0 & 2A \\ 0 & I_n & 0 \\ 0 & 0 & I_n+A \end{bmatrix}\xrightarrow{P_4}\begin{bmatrix} A-A^2 & 0 & 2A \\ 0 & I_n & 0 \\ -\frac{1}{2}(A-A^2) & 0 & I_n \end{bmatrix}$$

$$\xrightarrow{P_5}\begin{bmatrix} A-A^3 & 0 & 0 \\ 0 & I_n & 0 \\ -\frac{1}{2}(A-A^2) & 0 & I_n \end{bmatrix}\xrightarrow{Q_4}\begin{bmatrix} A-A^3 & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix}=N$$

and we have:

$$rank\,M = rank\,A + rank(I_n - A) + rank(I_n + A) = rank\,N = rank(A-A^3)+2n$$

and from c) we obtain: $rank(A-A^3)=0$, hence $A-A^3=0$.

The transform matrices on lines and column are:

$$P_1=\begin{bmatrix} I_n & 0 & 0 \\ I_n & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix},\ P_2=\begin{bmatrix} I_n & -A & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix},\ P_3=\begin{bmatrix} I_n & 0 & A \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix},$$

$$P_4=\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ -\frac{1}{2}I_n & 0 & I_n \end{bmatrix},\ P_5=\begin{bmatrix} I_n & 0 & -2A \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix},\ Q_1=\begin{bmatrix} I_n & I_n & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix},$$

$$Q_2=\begin{bmatrix} I_n & 0 & 0 \\ -A & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix},\ Q_3=\begin{bmatrix} I_n & 0 & I_n \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix},\ Q_4=\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ \frac{1}{2}(A-A^2) & 0 & I_n \end{bmatrix}$$

and we get: $N = P_5\cdot P_4\cdot P_3\cdot P_2\cdot P_1\cdot M\cdot Q_1\cdot Q_2\cdot Q_3\cdot Q_4$.

f) $\rightarrow$ a)

$$M=\begin{bmatrix} A+A^2 & 0 \\ 0 & A-A^2 \end{bmatrix}\xrightarrow{P_1}\begin{bmatrix} A+A^2 & A-A^2 \\ 0 & A-A^2 \end{bmatrix}$$

$$\xrightarrow{Q_1}\begin{bmatrix} A+A^2 & 2A \\ 0 & A-A^2 \end{bmatrix}\xrightarrow{Q_2}\begin{bmatrix} A & 2A \\ -\frac{1}{2}(A^2-A^3) & A-A^2 \end{bmatrix}$$

$$\xrightarrow{Q_3}\begin{bmatrix} A & 0 \\ -\frac{1}{2}(A^2-A^3) & A-A^3 \end{bmatrix}\xrightarrow{P_2}\begin{bmatrix} A & 0 \\ 0 & A-A^3 \end{bmatrix}=N.$$

We have: $rank\,M = rank(A+A^2)+rank(A-A^2) = rank\,N = rank\,A + rank(A-A^3)$ and from f) it follows $rank(A-A^3)=0$, hence $A-A^3=0$.

The elementary matrices corresponding to the elementary operations are:

$$P_1 = \begin{bmatrix} I_n & I_n \\ 0 & I_n \end{bmatrix},\ Q_1 = \begin{bmatrix} I_n & I_n \\ 0 & I_n \end{bmatrix},\ Q_2 = \begin{bmatrix} I_n & 0 \\ -\frac{1}{2}A & I_n \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} I_n & -2I_n \\ 0 & I_n \end{bmatrix},\ P_2 = \begin{bmatrix} I_n & 0 \\ \frac{1}{2}(A-A^2) & I_n \end{bmatrix}$$

and we have: $M = P_2 \cdot P_1 \cdot M \cdot Q_1 \cdot Q_2 \cdot Q_3$.

Remark 1.1. The last characterization (f) from statement Theorem 1.1 without proof can be found in [2].

2. CHARACTERIZATION OF IDEMPOTENT AND INVOLUTORY MATRICES

Recall that $A \in M_n(\mathbf{C})$ is an idempotent matrix if $A^2 = A$ and A is an involutory matrix if $A^2 = I_n$.

Theorem 2.1. [1] *For a matrix* $A \in M_n(\mathbf{C})$ *the following assertions are equivalent:*
a) $A^2 = A$;
b) $rank\, A + rank(I_n - A) = n$.

Proof. a) $\to$ b): If $A^2 = A$ then the eigenvalues of A satisfy $\lambda^2 = \lambda$, then $\lambda \in \{0,1\}$.

The Jordan canonical form of A verifies $J_A^2 = J_A$ and then all Jordan cells have dimension 1, so J_A is a diagonal matrix

$$J_A = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix},$$

where k is the number of eigenvalues of A equal to 1.

We have: $rank\, A = rank\, P \cdot J_A \cdot P^{-1} = rank\, J_A = k$
and $rank(I_n - A) = rank\, P \cdot (I_n - J_A) \cdot P^{-1} = rank(I_n - J_A) = n - k$,
hence $rank\, A + rank(I_n - A) = n$.

b) $\to$a): We consider the block matrix

$$M = \begin{bmatrix} A & 0 \\ 0 & I_n - A \end{bmatrix}$$

and we apply the elementary operations which preserve the rank

$$M \xrightarrow{P_1} \begin{bmatrix} A & 0 \\ A & I_n - A \end{bmatrix} \xrightarrow{Q_1} \begin{bmatrix} A & A \\ A & I_n \end{bmatrix}$$

$$\xrightarrow{P_2}\begin{bmatrix} A-A^2 & 0 \\ A & I_n \end{bmatrix}\xrightarrow{Q_2}\begin{bmatrix} A-A^2 & 0 \\ 0 & I_n \end{bmatrix}=N$$

and we have:

$$\begin{aligned} rank\,M &= rank\,A + rank(I_n - A) = rank\,N \\ &= rank(A-A^2) + rank\,I_n = rank(A-A^2)+n \end{aligned}$$

and from b) it follows $rank(A-A^2)=0$, hence $A-A^2=0$.

The matrices of elementary operations on rows P_1, P_2 and on columns Q_1, Q_2 are:

$$P_1=\begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix},\ P_2=\begin{bmatrix} I_n & -A \\ 0 & I_n \end{bmatrix},\ Q_1=\begin{bmatrix} I_n & I_n \\ 0 & I_n \end{bmatrix},\ Q_2=\begin{bmatrix} I_n & 0 \\ -A & I_n \end{bmatrix}$$

and we have: $N=P_2\cdot P_1\cdot M\cdot Q_1\cdot Q_2$.

Theorem 2.2. [3] *For a matrix* $A\in M_n(\mathbf{C})$ *the following statements are equivalent:*

a) $A^2=I_n$;

b) $rank(I_n-A)+rank(I_n+A)=n$.

Proof. If $A^2=I_n$ then the eigenvalues of A satisfies the relation $\lambda^2=1$, hence $\lambda\in\{-1,1\}$. The canonical form J_A verifies the same relation $J_A^2=I_n$ and it is a diagonal matrix:

$$J_A=\begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix},$$

where k is the number of the eigenvalues equal to 1 and $n-k$ is the number of eigenvalues equal to -1.

We have:

$$rank(I_n-A)=rank(I_n-J_A)=n-k$$

and

$$rank(I_n+A)=rank(I_n+J_A)=k,$$

therefore:

$$rank(I_n-A)+rank(I_n+A)=k+n-k=n.$$

For the converse implication we consider the block matrix

$$M=\begin{bmatrix} I_n-A & 0 \\ 0 & I_n+A \end{bmatrix}$$

in which we consider the following elementary operations:

$$M \xrightarrow{Q_1} \begin{bmatrix} I_n - A & I_n - A \\ 0 & I_n + A \end{bmatrix} \xrightarrow{Q_2} \begin{bmatrix} I_n - A^2 & I_n - A \\ A + A^2 & I_n + A \end{bmatrix} \xrightarrow{P_2} \begin{bmatrix} I_n - A^2 & I_n - A \\ I_n + A & 2I_n \end{bmatrix}$$

$$\xrightarrow{P_2} \begin{bmatrix} \frac{1}{2}(I_n - A^2) & 0 \\ I_n + A & 2I_n \end{bmatrix} \longrightarrow \begin{bmatrix} \frac{1}{2}(I_n - A^2) & 0 \\ 0 & I_n \end{bmatrix} = N .$$

We have:

$$rank\, M = rank(I_n - A) + rank(I_n + A) = rank\, N$$
$$= rank\, \frac{1}{2}(I_n - A^2) + rank(2I_n) = rank(I_n - A^2) + n$$

and from b) it follows $rank(I_n - A^2) = 0$, hence $I_n - A^2 = 0$.

The matrices P_1, P_2 of the elementary operations on rows and Q_1, Q_2, Q_3 of the elementary operations on columns are:

$$P_1 = \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix},\ P_2 = \begin{bmatrix} I_n & -\frac{1}{2}(I_n - A) \\ 0 & I_n \end{bmatrix},\ Q_1 = \begin{bmatrix} I_n & I_n \\ 0 & I_n \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} I_n & 0 \\ I_n & I_n \end{bmatrix},\ Q_3 = \begin{bmatrix} I_n & 0 \\ -\frac{1}{2}(I_n + A) & I_n \end{bmatrix}$$

and $N = P_2 \cdot P_1 \cdot M \cdot Q_1 \cdot Q_2 \cdot Q_3$.

3. CHARACTERIZATION USING THE RANK OF SOME MATRICES HAVING ANNIHILATOR POLYNOMIALS OF THE FORM $f(x) = x^k - x^p$

We present below three results whose proofs can be done using the same ideas and methods used in previous theorems.

Theorem 3.1. *For a matrix* $A \in M_n(\mathbf{C})$ *the following statements are equivalent:*
a) $A^2 = A^3$;
b) $rank\, A^2 + rank(I_n - A) = n$.

Theorem 3.2. *For a matrix* $A \in M_n(\mathbf{C})$ *the following statements are equivalent:*
a) $A^3 = A^5$;
b) $rank\, A^3 + rank(I_n - A^2) = n$;
c) $rank(I_n - A) + rank(A^3 + A^4) = n$;
d) $rank(I_n + A) + rank(A^3 - A^4) = n$;
e) $rank\, A^3 + rank(I_n - A) + rank(I_n + A) = 2n$;
f) $rank(A - A^2) + rank(A^3 + A^4) = rank\, A$;

g) $rank(A+A^2)+rank(A^3-A^4)=rank\,A$;

h) $rank(A^3+A^4)+rank(A^3-A^4)=rank\,A^3$.

Theorem 3.3. *If* p,q *are positive integers and* $A\in M_n(\mathbf{C})$ *then the following statements are equivalent:*

a) $A^p=A^{p+q}$;

b) $rank\,A^p+rank(A^q-I_n)=n$.

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