ORIGINAL PAPER

ON FRACTIONAL INTEGRAL INEQUALITIES INVOLVING THE SAIGO'S FRACTIONAL INTEGRAL OPERATORS

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Abstract. *New classes of integral inequalities are established in this paper by using a family of* n *positive functions,* $(n \in \mathbb{N})$ *with the help of Saigo's fractional integral operator. Some consequent results and special cases of the main results are also pointed out.*

Keywords: *Integral inequalities, fractional integral operators, Saigo's fractional integral operator.*

1. INTRODUCTION

Inequality plays an important in the field of physical problems like in electrostatics, symmetric potential theory and elasticity. In recent years many authors worked and established certain integral inequalities by using well known fractional integral operators; see for example [1-9].

The aim of this paper is to generalize some classical inequalities. By using Saigo's fractional integral operator, we generate new classes of integral inequalities using a family of n positive functions defined on $[a\ , b]$.

First we give necessary definitions and mathematical preliminaries of fractionalcalculus operators which are used in our analysis.

Definition 1. A real valued function $f(t)(t>0)$ is said to be in the space $\mathbb{C}_\mu(\mu \in \Re$, if there exists a real number $p>\mu$ such that $ft=tp\phi(t)$; where $\phi(t)\in\mathbb{C}(0,\infty)$.

Definition 2. Let $\alpha>0$ and $\beta, \eta \in \Re$, then the Saigo fractional integral $I_{0,t}^{\alpha,\beta,\eta}$ of order $\alpha$for a real valued continuous function $f(t)$is defined by [10]

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\tau}{t}\right) f(\tau) d\tau, \tag{1}$$

where the function ${}_2F_1\ (-)$ appearing as a kernel of the operator (1) is the Gaussian hypergeometric function defined by

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$$ {}_2F_1\ (a,b;c;t)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n t^n}{n!(c)_n}, \tag{2}$$

where $(a)_n$ is Pochhammer symbol given as

$$(a)_n=a(a+1)\dots(a+n-1),(a)_0=1. \tag{3}$$

2. MAIN RESULTS

In this section, we established two classes of fractional integral inequalities for the synchronous functions involving the Saigo fractional integral operator (1). The first class is given by the following two theorems:

Theorem 1. *Let* $(f_i)_{i=1,\dots,n}$ *are* n *positive continuous and decreasing functions on*$[a\,,b]$, *then*

$$\frac{I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right]}{I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]}\geq\frac{I_{0,t}^{\alpha,\beta,\eta}\left[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right]}{I_{0,t}^{\alpha,\beta,\eta}\left[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]}, \tag{4}$$

for $a<t\leq b,\alpha>\max\{0,-\beta\},\beta<1,\ \beta-1<\eta<0,\ \delta>0,\zeta\geq\gamma_p>0,$ *where* p *is fixed integer in* $\{1,2,\dots,n\}$.

Proof. Since $(f_i)_{i=1,\dots,n}$ are n positive continuous and decreasing functions on $[a\,,b]$ then we have

$$\left((\rho-a)^{\delta}-(\tau-a)^{\delta}\right)\left(f_p^{\zeta-\gamma_p}(\tau)-f_p^{\zeta-\gamma_p}(\rho)\right)\geq 0,$$

which implies that

$$(\rho-a)^{\delta}f_p^{\zeta-\gamma_p}(\tau)+(\tau-a)^{\delta}f_p^{\zeta-\gamma_p}(\rho)\geq\ (\tau-a)^{\delta}f_p^{\zeta-\gamma_p}(\tau)+(\rho-a)^{\delta}f_p^{\zeta-\gamma_p}(\rho), \tag{5}$$

for any fixed $p\in\{1,\dots,n\}$ and for any $\zeta\geq\gamma_p>0,\delta>0,\tau,\rho\in[a,t];a<t\leq b$.
Consider

$$\text{Ɲ}N_p(t,\tau)=\frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\left(t-\tau\ \ \right)^{\alpha-1}\ \ {}_2F_1\ \left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\ \prod_{i=1}^{n} f_i^{\gamma_i}(\tau). \tag{6}$$

We observe each factor of the above function is positive in view of the conditions stated with Theorem 1, and hence the function $\text{Ɲ}N_p(t,\tau)$remains positive, for all $\tau\in(0,t)\ (t>\ 0)$.

Multiplying both sides of (5) byƝ $N_p(t,\tau)$ (where $N\text{Ɲ}_p(t,\tau)$ is given by (6)) and integrating with respect to τ from 0 to t, and using operator (1), we get

$$(\rho-a)^{\delta} I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] + f_p^{\zeta-\gamma_p}(\rho) I_{0,t}^{\alpha,\beta,\eta}\left[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] \geq$$
$$(\rho-a)^{\delta} f_p^{\zeta-\gamma_p}(\rho) I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] + I_{0,t}^{\alpha,\beta,\eta}\left[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right]. \quad (7)$$

Now multiplying both sides of (7) by $\daleth N_p(\tau,\rho)$ where $\daleth N_p(\tau,\rho)$ is given in (6) and integrating with respect to ρ from 0 to t, and using operator (1), we get

$$I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] \geq$$
$$I_{0,t}^{\alpha,\beta,\eta}\left[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]. \quad (8)$$

This completes the proof of Theorem 1.

Theorem 2. *Let $(f_i)_{i=1,\ldots,n}$ are n positive continuous and decreasing functions on $[a\,,b]$, then*

$$I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\omega,\xi,\psi}\left[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] +$$
$$I_{0,t}^{\omega,\xi,\psi}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] \; I_{0,t}^{\alpha,\beta,\eta}\left[(t-a)^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] \geq$$
$$I_{0,t}^{\omega,\xi,\psi}\left[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] +$$
$$I_{0,t}^{\alpha,\beta,\eta}\left[(t-a)^{\delta}\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\omega,\xi,\psi}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right], \quad (9)$$

for $a < t \leq b, \alpha > \max\{0,-\beta\}, \omega > \max\{0,-\xi\}, \beta < 1, \xi < 1,\ \xi - 1 < \psi < 0,$
$\zeta \geq \gamma_p > 0,$ *where* p *is fixed integer in* $\{1,2,\ldots,n\}$.

Proof. On multiplying both side of equation (7) by

$$\frac{t^{-\omega-\xi}}{\Gamma(\omega)}\left(t-\rho\ \right)^{\omega-1} {}_2F_1\ \left(\omega+\xi,-\psi;\omega;1-\frac{\rho}{t}\right)\prod_{i=1}^{n} f_i^{\gamma_i}(\rho\ \ ),\ \ \left(\rho\in(0,\mathrm{t});\quad t>0\right), \quad (10)$$

(which remains positive under the conditions stated with Theorem 2), and then integrating the resulting inequality so obtained, with respect to ρ from 0 to t, we get the desired result.

Remark 1. The inequality (4) and (9) are reversed if the functions $(f_i)_{i=1,\ldots,n}$ is increasing on $[a,b]$.

Remark 2. Applying Theorem 2 for $\omega = \alpha, \xi = \beta, \rho = \tau and \psi = \eta$ we obtain Theorem 1

Another class of fractional integral inequalities which generalizes the above theorems is described in the following theorems.

Theorem 3. *Let $(f_i)_{i=1,\dots,n}$ and g be positive continuous functions on $[a\,,b]$ such that g is increasing and $(f_i)_{i=1,\dots,n}$ are decreasing on [a ,b] then:*

$$\frac{I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]}{I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]} \geq 1. \tag{11}$$

Provided $a < t \leq b, \alpha > \max\{0,-\beta\}, \beta < 1,\ \beta-1 < \eta < 0,\ \delta > 0, \zeta \geq \gamma_p > 0$, where p is fixed integer in $\{1,2,\dots,n\}$.

Proof. Using condition of Theorem 3 we can write

$$\left(g^{\delta}(\rho) - g^{\delta}(\tau)\right)\left(f_p^{\zeta-\gamma_p}(\tau) - f_p^{\zeta-\gamma_p}(\rho)\right) \geq 0, \tag{12}$$

for all $p = 1,2,\dots,n, \delta > 0, \zeta \geq \gamma_p > 0, \rho,\tau \in [a,t], a < t \leq b$.

Now, let's consider the quantity

$$L_p(t,\tau) = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}\ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right) \prod_{i=1}^{n} f_i^{\gamma_i}(\tau)\left(\left(g^{\delta}(\rho)-g^{\delta}(\tau)\right)\right)\left(f_p^{\zeta-\gamma_p}(\tau)-f_p^{\zeta-\gamma_p}(\rho)\right). \tag{13}$$

It is clear that

$$L_p(t,\tau) \geq 0. \tag{14}$$

Therefore,

$$0 \leq \int_0^t L_p(t,\tau)d\tau = g^{\delta}(\rho) I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] + f_p^{\zeta-\gamma_p}(\rho) I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] - I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] - g^{\delta}(\rho) f_p^{\zeta-\gamma_p}(\rho) I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]. \tag{15}$$

Consequently,

$$I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] \geq I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]. \tag{16}$$

Theorem 3 is thus proved.

We also give the following results:

Theorem 4. *Let $(f_i)_{i=1,\dots,n}$ and g are positive continuous functions on $[a\,,b]$, such that g is increasing and $(f_i)_{i=1,\dots,n}$ are decreasing on $[a\,,b]$ then:*

$$I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\omega,\xi,\psi}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]+$$

$$I_{0,t}^{\omega,\xi,\psi}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] \ I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]\geq$$

$$I_{0,t}^{\omega,\xi,\psi}\left[g(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]+$$

$$I_{0,t}^{\alpha,\beta,\eta}\left[g(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\omega,\xi,\psi}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right], \tag{17}$$

for $a < t \leq b, \alpha > \max\{0,-\beta\}, \omega > \max\{0,-\xi\}, \beta < 1, \xi < 1,\ \xi - 1 < \psi < 0,$ $\zeta \geq \gamma_p > 0$, *where* p *is fixed integer in* $\{1,2,\ldots,n\}$.

Proof. Multiplying both side of (12) by

$$\frac{t^{-\omega-\xi}}{\Gamma(\omega)}(t-\rho)^{\omega-1}\ {}_2F_1\left(\omega+\xi,-\psi;\omega;1-\frac{\rho}{t}\right)\ \prod_{i=1}^{n} f_i^{\gamma_i}(\rho),\ \left(\rho\in(0,t);\quad t>0\right), \tag{18}$$

and then integrating with respect to ρ from 0 to t, we get the desired result.

Theorem 5. *Let* $(f_i)_{i=1,\ldots,n}$ *and* g *be positive continuous functions on* $[a\,,b]$. *Suppose that for any fixed* $p \in \{1,2,\ldots,n\}$, $\left(f_p^{\delta}(\tau)g^{\delta}(\rho) - f_p^{\delta}(\rho)g^{\delta}(\tau)\right)\left(f_p^{\zeta-\gamma_p}(\tau) - f_p^{\zeta-\gamma_p}(\rho)\right) \geq 0; \delta > 0, \alpha > 0, \zeta \geq \gamma_p > 0, \tau,\rho \in [a\,,t], t \in]a\,,b]$ *then we have*

$$\frac{I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta+\delta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]}{I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[f_p^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]} \geq 1. \tag{19}$$

Proof. The proof is quite similar to the Theorem 3, provided we replace the quantity

$$\left(g^{\delta}(\rho) - g^{\delta}(\tau)\right) by \left(f_p^{\delta}(\tau)g^{\delta}(\rho) - f_p^{\delta}(\rho)g^{\delta}(\tau)\right).$$

Theorem 6. *Let* $(f_i)_{i=1,\ldots,n}$ *and* g *be positive continuous functions on* $[a\,,b]$. *Suppose that for any fixed* $p \in \{1,2,\ldots,n\}$, $\left(f_p^{\delta}(\tau)g^{\delta}(\rho) - f_p^{\delta}(\rho)g^{\delta}(\tau)\right)\left(f_p^{\zeta-\gamma_p}(\tau) - f_p^{\zeta-\gamma_p}(\rho)\right) \geq 0;\delta > 0, \alpha > 0, \zeta \geq \gamma_p > 0, \tau,\rho \in [a\,,t], t \in]a\,,b]$. *Then the inequality*

$$I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta+\delta}(t)\right] I_{0,t}^{\omega,\xi,\psi}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]+$$
$$I_{0,t}^{\omega,\xi,\psi}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta+\delta}(t)\right]\ I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]\geq$$
$$I_{0,t}^{\alpha,\beta,\eta}\left[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\omega,\xi,\psi}\left[f_p^{\delta}\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]+$$
$$I_{0,t}^{\omega,\xi,\psi}\left[g^{\delta}(t)\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_{0,t}^{\alpha,\beta,\eta}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right], \tag{20}$$

holds true.

Proof. The proof is quite similar to Theorem 4, provided we replace the quantity

$$\left(g^{\delta}(\rho) - g^{\delta}(\tau)\right) by \left(f_p^{\delta}(\tau)g^{\delta}(\rho) - f_p^{\delta}(\rho)g^{\delta}(\tau)\right).$$

3. SPECIAL CASES

The operator $I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}$ includes both the Riemann-Liouville and Erdélyi-Kober fractional integral operators by the following relationships:

$$R^{\alpha}\{f(t)\} = I_{0,\mathrm{t}}^{\alpha,-\alpha,\eta}\{f(t)\} = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau (\alpha > 0) \tag{21}$$

and

$$I^{\alpha,\eta}\{f(t)\} = I_{0,\mathrm{t}}^{\alpha,0,\eta}\{f(t)\} = \frac{t^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^t \left(t-\tau\right)^{\alpha-1} \tau^{\eta} f(\tau)d\tau, (\alpha > 0, \eta \in \Re). \tag{22}$$

Also for $f(t) = t^{\mu}$ in (1) we get the known formula [10]:

$$I_{0,t}^{\alpha,\beta,\eta}\{t^{\mu}\} = \frac{\Gamma(\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\mu+1+\alpha+\eta)} t^{\mu-\beta}, \tag{23}$$

where $\alpha > 0,\ \min(\mu, \mu-\beta+\eta) > -1, t > 0.$

It is interesting to observe that, if we replace β by $-\alpha$ and make use of relation (21), our main results correspond to the known results of Dahmani [6]. Further, by setting $\beta = 0$ and using the relation (22), the inequalities presented in the main theorems reduces to the known fractional integral inequalities involving Erdélyi-Kober fractional integral operators.

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