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GROUPLIKE ELEMENTS FOR TRIGONOMETRIC AND HYPERBOLIC COALGEBRAS

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Abstract. *In this paper we determine all grouplike elements of some some classes of coalgebras over a field* k, *as well as: trigonometric coalgebras and hyperbolic coalgebras. Also we prove that matrix coalgebra* $M^{C}(2,k)$ *doesn't have any grouplike element.*

Keywords: *grouplike elements, hyperbolic coalgebras, trigonometric coalgebras.*

1. GROUPLIKE ELEMENTS FOR THE TRIGONOMETRIC COALGEBRA

In the next paragraph we first present the construction of a coalgebra, namely trigonometric coalgebra, starting from the well known trigonometric formulas:

$$\sin(x+y)=\sin x\cdot\cos y+\cos x\cdot\sin y$$

$$\cos(x+y)=\cos x\cdot\cos y-\sin x\cdot\sin y.$$

Because the trigonometric function *sin* and *cos* are linear independent, we can consider a two dimension k – linear space T with basis $\{\sin,\cos\}$, which becomes a coalgebra with the comultiplication $\Delta_T : T\otimes T\to T$ and the counit $\varepsilon_T : T\to k$:

$$\Delta(\sin)(x+y):=\sin(x+y) \text{ and } \Delta(\cos)=\cos\otimes\cos-\sin\otimes\sin$$

and also:

$$\varepsilon(\sin)=0 \text{ and } \varepsilon(\cos)=1.$$

In general, let's consider C_t be a two dimension k – linear space with basis $\{s,c\}$, which becomes a coalgebra with the comultiplication:

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$$\Delta_t : C_t \otimes C_t \to C_t,\ \Delta_t(s) = s \otimes c + c \otimes s \text{, respectively } \Delta_t(c) = c \otimes c - s \otimes s,$$

and the counit:

$$\varepsilon_t : C_t \to k,\ \varepsilon_t(s) = 0 \text{ , respectively } \varepsilon_t(c) = 1.$$

In this way, the triplet $(C_t, \Delta_t, \varepsilon_t)$ is a coalgebra, named *the trigonometric coalgebra* over the field *k.*

It is very easy to prove that $(C_t, \Delta_t, \varepsilon_t)$ is in fact a coalgebra. For this we have:

$$(I \otimes \Delta_t) \circ \Delta_t(s) = (I \otimes \Delta_t)(s \otimes c + c \otimes s) = (I \otimes \Delta_t)(s \otimes c) + (I \otimes \Delta_t)(c \otimes s) =$$
$$= s \otimes \Delta_t(c) + c \otimes \Delta_t(s) = s \otimes (c \otimes c - s \otimes s) + c \otimes (s \otimes c + c \otimes s) =$$
$$= s \otimes c \otimes c - s \otimes s \otimes s + c \otimes s \otimes c + c \otimes c \otimes s = (\Delta_t \otimes I) \circ \Delta_t(s).$$

In a similar way, we have:

$$(I \otimes \Delta_t) \circ \Delta_t(c) = (\Delta_t \otimes I) \circ \Delta_t(s) = c \otimes c \otimes c - s \otimes s \otimes c - s \otimes c \otimes s - c \otimes s \otimes s,$$

and so, the application Δ_t is a comultiplication. Similar, we obtain:

$$(I \otimes \varepsilon) \circ \Delta_t(s) = (I \otimes \varepsilon)(s \otimes c + c \otimes s) = (I \otimes \varepsilon)(s \otimes c) + (I \otimes \varepsilon)(c \otimes s) =$$
$$= s \otimes \varepsilon(c) + c \otimes \varepsilon(s) = s \otimes 1 + c \otimes 0 = s + 0 = s \text{ and}$$
$$(\varepsilon_t \otimes I) \circ \Delta_t(s) = (\varepsilon_t \otimes I)(s \otimes c + c \otimes s) = (\varepsilon_t \otimes I)(s \otimes c) + (\varepsilon_t \otimes I)(c \otimes s) =$$
$$= \varepsilon_t(s) \otimes c + \varepsilon_t(c) \otimes s = 0 \otimes c + 1 \otimes s = 0 + s = s \text{, from where we have:}$$
$$(I \otimes \varepsilon_t) \circ \Delta_t(s) = (\varepsilon_t \otimes I) \circ \Delta_t(s) = s \text{, and also:}$$
$$(I \otimes \varepsilon_t) \circ \Delta_t(c) = (\varepsilon_t \otimes I) \circ \Delta_t(c) = c.$$

All these relations prove that ε_t is a counit on C_t.

Remark. The property of Δ_t to be a comultiplication can be translated in the following trigonometric formulas:

$$\sin(x+y+z)=\sin x\cdot\cos y\cdot\cos z-\sin x\cdot\sin y\cdot\sin z+\\+\cos x\cdot\sin y\cdot\cos z+\cos x\cdot\cos y\cdot\sin z$$

$$\cos(x+y+z)=\cos x\cdot\cos y\cdot\cos z-\sin x\cdot\sin y\cdot\cos z-\\-\sin x\cdot\cos y\cdot\sin z-\cos x\cdot\sin y\cdot\sin z$$

Now, remind that for a coalgebra *C*, we call a grouplike element, an element $g\neq 0$ with $\Delta(g)=g\otimes g$ and $\varepsilon(g)=1$. Also, it is well known that the set of all grouplike elements of a coalgebra, denoted by $G(C)$, is a set of linear independent elements.

To find all the grouplike elements of the trigonometric coalgebra, let consider an arbitrary element $x\in(C_t,\Delta_t,\varepsilon_t)$. Because C_t is a k – linear space with basis $\{s,c\}$, then does exist two scalars *a* and *b* from *k* such that:

$$x=a\cdot s+b\cdot c.$$

We obtain:

$$\Delta_t(x)=\Delta_t(a\cdot s+b\cdot c)=\Delta_t(a\cdot s)+\Delta_t(b\cdot c)=a\Delta_t(s)+b\Delta_t(c)=\\=a(s\otimes c+c\otimes s)+b(c\otimes c-s\otimes s)=a\cdot s\otimes c+a\cdot c\otimes s+b\cdot c\otimes c-b\cdot s\otimes s$$

and:

$$x\otimes x=(a\cdot s+b\cdot c)\otimes(a\cdot s+b\cdot c)=a\cdot s\otimes(a\cdot s+b\cdot c)+b\cdot c\otimes(a\cdot s+b\cdot c)=\\=a\cdot s\otimes a\cdot s+a\cdot s\otimes b\cdot c+b\cdot c\otimes a\cdot s+b\cdot c\otimes b\cdot c=\\=a^2\cdot s\otimes s+ab\cdot s\otimes c+ba\cdot c\otimes s+b^2\cdot c\otimes c.$$

Also, because C_t is a space of basis $\{s,c\}$, we obtain that $C_t\otimes C_t$ is also a linear space with the basis $\{s\otimes s,s\otimes c,c\otimes s,c\otimes c\}$. The condition that x is a grouplike element of C_t is equivalent to $\Delta_t(x)=x\otimes x$, and so we obtain the following relations:

$$\begin{cases}ab=a\\a^2=-b\\b=b^2\end{cases}$$

But $\varepsilon_t(x) = 1$, then: $\varepsilon_t(x) = \varepsilon_t(a \cdot s + b \cdot c) = a \cdot \varepsilon_t(s) + b \cdot \varepsilon_t(c) = a \cdot 0 + b \cdot 1 = b = 1$

We obtain $b = 1$ and $a^2 = -1$.

If k = **R**, then a does not exist, in this case the trigonometric coalgebra doesn't have any grouplike element. But if k = **C**, then $a_1 = i$ or $a_2 = -i$, and in this case the coalgebra C_t have two grouplike elements: $x_1 = is + c$ and $x_2 = is + c$.

***Consequences*:**

1. The trigonometric coalgebra over the field of real numbers **R** is a simple coalgebra, being of dimension two and not having any subcoalgebra of dimension one, this because there are no grouplike elements.
2. In the case of trigonometric coalgebra over the field of complex numbers, C_t have two grouplike elements, and so two subcoalgebras of dimension one which are in bijective correspondence with the set $G(C_t) = \{x_1, x_2\}$. In this case C_t is not a simple coalgebra.

2. GROUPLIKE ELEMENTS FOR THE HYPERBOLIC COALGEBRA

Let's remind two hyperbolic formulas:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} \text{ and } \sinh(x) = \frac{e^x + e^{-x}}{2} = \frac{e^{2x} + 1}{2e^x}.$$

It is easy to prove that:

$$\sinh(x + y) = \sinh(x) \cdot \cosh(y) + \cosh(x) \cdot \sinh(y)$$

$$\cosh(x + y) = \cosh(x) \cdot \cosh(y) + \sinh(x) \cdot \sinh(y).$$

In a similar way like in the case of the trigonometric coalgebra, we can define a new one, named *the hyperbolic coalgebra* C_h. This is a k – linear space with basis $\{\sinh, \cosh\}$ on which we can define a comultiplication and a counit:

$$\Delta_h(\sinh) = \sinh \otimes \cosh + \cosh \otimes \sinh \text{ and } \Delta_h(\cosh) = \cosh \otimes \cosh + \sinh \otimes \sinh ,$$

$$\varepsilon_h(\sinh) = 0 \text{ and } \varepsilon_h(\cosh) = 1.$$

Because the hyperbolic functions *sinh* and *cosh* are linear independent, we can generalize this to a coalgebra C_h, a k – linear space of two dimension with basis $\{sh, ch\}$, on which we define two linear applications:

$$\Delta_h : C_h \to C_h \otimes C_h \text{, by } \Delta_h(sh) = sh \otimes ch + ch \otimes sh \text{ and } \Delta_h(ch) = ch \otimes ch + sh \otimes sh$$

$$\text{and } \varepsilon_h : C_h \to k \text{ , } \varepsilon_h(sh) = 0 \text{ and } \varepsilon_h(ch) = 1,$$

in this way we obtain a new coalgebra, $(C_h, \Delta_h, \varepsilon_h)$ called *the hyperbolic coalgebra.*

It is easy to prove that Δ_h and ε_h are the comultiplication and the counit of this coalgebra, means we have:

$$\begin{aligned}(I \otimes \Delta_h) \otimes \Delta_h(sh) = (\Delta_h \otimes I) \otimes \Delta_h(sh) &= sh \otimes ch \otimes ch + sh \otimes sh \otimes sh + \\ &+ ch \otimes sh \otimes ch + ch \otimes ch \otimes ch \\ (I \otimes \Delta_h) \otimes \Delta_h(ch) = (\Delta_h \otimes I) \otimes \Delta_h(ch) &= ch \otimes ch \otimes ch + sh \otimes sh \otimes ch + \\ &+ ch \otimes sh \otimes sh + sh \otimes ch \otimes sh\end{aligned}$$

and:

$$\begin{aligned}(I \otimes \varepsilon_h) \otimes \Delta_h(sh) = (\varepsilon_h \otimes I) \otimes \Delta_h(sh) = sh \\ (I \otimes \varepsilon_h) \otimes \Delta_h(ch) = (\varepsilon_h \otimes I) \otimes \Delta_h(ch) = ch.\end{aligned}$$

Now, let's consider $x \in (C_h, \Delta_h, \varepsilon_h)$ an arbitrary element. Then, there exist two scalars *a* and *b* from *k* such that:

$$x = a \cdot sh + b \cdot ch.$$

Using the comultiplication Δ_h we obtain:

$$\begin{aligned}\Delta_h(x) &= \Delta_h(a\cdot sh + b\cdot ch) = \Delta_h(a\cdot sh) + \Delta_h(b\cdot ch) = a\Delta_h(sh) + b\Delta_h(ch) = \\ &= a(sh\otimes ch + ch\otimes sh) + b(ch\otimes ch - sh\otimes sh) = \\ &= a\cdot sh\otimes ch + a\cdot ch\otimes sh + b\cdot ch\otimes ch - b\cdot sh\otimes sh\end{aligned}$$

and because of the bilinear tensor product we have:

$$\begin{aligned}x\otimes x &= (a\cdot sh + b\cdot ch)\otimes(a\cdot sh + b\cdot ch) = \\ &= a\cdot sh\otimes(a\cdot sh + b\cdot ch) + b\cdot ch\otimes(a\cdot sh + b\cdot ch) = \\ &= a\cdot sh\otimes a\cdot sh + a\cdot sh\otimes b\cdot ch + b\cdot ch\otimes a\cdot sh + b\cdot ch\otimes b\cdot ch = \\ &= a^2\cdot sh\otimes sh + ab\cdot sh\otimes ch + ba\cdot ch\otimes sh + b^2\cdot ch\otimes ch.\end{aligned}$$

But C_h is a k vector space of basis $\{sh, ch\}$, so the tensor coalgebra $C_h\otimes C_h$ is a space of basis $\{sh\otimes sh, sh\otimes ch, ch\otimes sh, ch\otimes ch\}$, and so x is a grouplike element if $\Delta_h(x) = x\otimes x$, which is equivalent to:

$$\begin{cases} ab = a \\ a^2 = b \\ b = b^2 \end{cases}$$

Much more: $\varepsilon_h(x) = 1$, and so we obtain:

$$\varepsilon_h(x) = \varepsilon_h(a\cdot s + b\cdot c) = a\cdot\varepsilon_h(s) + b\cdot\varepsilon_h(c) = a\cdot 0 + b\cdot 1 = b = 1$$

Finaly we have $b = 1$ and $a^2 = 1$, with two solutions $a_1 = 1$ and $a_2 = -1$, for k = **R** and also for k = **C**. This shows us that the hyperbolic coalgebra C_h have two grouplike elements: $x_1 = sh + ch$ and $x_2 = sh - ch$, which are linear independent.

Consequence. The hyperbolic coalgebra C_h is not a simple coalgebra (having two grouplike elements, this coalgebra admit two subcoalgebras of dimension one).

3. GROUPLIKE ELEMENTS OF MATRIX COALGEBRA $M^C(2,k)$

Proposition. The matrix coalgebra $M^C(2,k)$ doesn't have any grouplike element.

Proof: The matrix coalgebra $M^C(2,k)$ is a k vector space of dimension $n^2=4$ with basis $\{e_{11},e_{12},e_{21},e_{22}\}$. Let $x\in M^C(2,k)$ be an arbitrary element. Then:

$$x=a_{11}e_{11}+a_{12}e_{12}+a_{21}e_{21}+a_{22}e_{22},\quad a_{ij}\in k, i,j=\overline{1,2}.$$

Let's suppose that x is a grouplike element, this is equivalent to $\Delta(x)=x\otimes x$ and $\varepsilon(x)=1$. We obtain:

$$\begin{aligned}\Delta(x)=&\,a_{11}e_{11}\otimes e_{11}+a_{11}e_{12}\otimes e_{21}+a_{12}e_{11}\otimes e_{12}+a_{12}e_{12}\otimes e_{22}+\\&+a_{21}e_{21}\otimes e_{11}+a_{21}e_{22}\otimes e_{21}+a_{22}e_{21}\otimes e_{12}+a_{22}e_{22}\otimes e_{22}\end{aligned}$$

and

$$\begin{aligned}x\otimes x=&\,a_{11}^2e_{11}\otimes e_{11}+a_{11}a_{12}e_{11}\otimes e_{12}+a_{11}a_{21}e_{11}\otimes e_{21}+a_{11}a_{22}e_{11}\otimes e_{22}+\\&+a_{12}a_{11}e_{12}\otimes e_{11}+a_{12}^2e_{12}\otimes e_{12}+a_{12}a_{21}e_{12}\otimes e_{21}+a_{12}a_{22}e_{12}\otimes e_{22}+\\&+a_{21}a_{11}e_{21}\otimes e_{11}+a_{21}a_{12}e_{21}\otimes e_{12}+a_{21}^2e_{21}\otimes e_{21}+a_{21}a_{22}e_{21}\otimes e_{22}+\\&+a_{22}a_{11}e_{22}\otimes e_{11}+a_{22}a_{12}e_{22}\otimes e_{12}+a_{22}a_{21}e_{22}\otimes e_{21}+a_{22}^2e_{22}\otimes e_{22}.\end{aligned}$$

From these, we have:

$$\begin{cases}a_{11}=a_{11}^2 & a_{11}a_{21}=0\\ a_{11}=a_{12}a_{21} & a_{11}a_{22}=0\\ a_{12}=a_{11}a_{12} & a_{12}a_{11}=0\\ a_{12}=a_{12}a_{22} & a_{12}^2=0\\ a_{21}=a_{21}a_{11} & a_{21}^2=0\\ a_{21}=a_{22}a_{21} & a_{21}a_{22}=0\\ a_{22}=a_{21}a_{12} & a_{22}a_{11}=0\\ a_{22}=a_{22}^2 & a_{22}a_{12}=0\end{cases}$$

with the solution :

$$a_{11} = a_{12} = a_{21} = a_{22} = 0 \Leftrightarrow x = 0,$$

These prove us that matrix coalgebra $M^C(2,k)$ doesn't have any grouplike element.

Consequence. Matrix coalgebra $M^C(2,k)$ doesn't have subcoalgebras of dimension one.

In general, the matrix coalgebra $M^C(n,k)$ doesn't have any grouplike element. The proof of this remark, using only the definition of the grouplike elements is not so simple and involve so many hard calculations. A simple poof of this we find in [1], ex. 1.4.17, where it is used the fact that $M^C(n,k)$ is the dual coalgebra of the matrix algebra $M_n(k)$, and so the set of all grouplike elements is $G(M^C(n,k)) = A\lg(M_n(k),k)$. Much more, there is no morfism of coalgebras $f : M_n(k) \to k$ with the kernel $\ker(f)$ coideal in $M_n(k)$, so there are only two situations: ker(f) = 0 and ker(f) = $M_n(k)$. In case $\ker(f) = 0$ we have that *f* is injective, impossible because the dimensions of $M_n(k)$ and *k*. Also, the case $\ker(f) = M_n(k)$ is impossible because $f(1) = 1$. Finally we have $G(M^C(n,k)) = 0$.

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