ORIGINAL PAPER

RAMANUJAN FORMULAS OF ODD ORDER FOR GAMMA FUNCTION

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Abstract. *We propose in this paper some approximation formulas of Ramanujan type of odd order.*

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1. INTRODUCTION

In almost all branches of science, we meet situations in which we are forced to estimate big factorials. One of the most known formulas used is the Stirling's formula:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$$

but more accurate results are obtained using the following formula due to Ramanujan [5]:

$$\Gamma(x+1) \sim \rho(x) := \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[6]{x^3+\frac{1}{2}x^2+\frac{1}{8}x+\frac{1}{240}}$$

Mortici [2] introduced the following new approximation formulas of Ramanujan's type:

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[4]{x^2+\frac{1}{3}x+\frac{1}{18}}$$

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[8]{x^4+\frac{2}{3}x^3+\frac{2}{9}x^2+\frac{11}{405}x-\frac{8}{1215}}$$

and

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[10]{x^5+\frac{5}{6}x^4+\frac{25}{72}x^3+\frac{89}{1296}x^2-\frac{95}{31104}x+\frac{2143}{1306368}}$$

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that are much stronger than Ramanujan formula.

The general method for establishing increasingly more accurate formulas of even order

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[2k]{x^k + \ldots}$$

is also presented by Mortici [3]. This result was also proven by Chen and Lin in [1].

We propose in this paper some approximation formulas of Ramanujan type of odd order, namely:

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[2s+1]{x^{\frac{2s+1}{2}} + a_1 x^{\frac{2s-1}{2}} + \ldots + a_s x^{\frac{1}{2}}} \quad (1)$$

This is an approximation formula at least in the general sense of an approximation formula $f(x) \sim g(x)$ for which $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$, since:

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{\sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[2s+1]{x^{\frac{2s+1}{2}} + a_1 x^{\frac{2s-1}{2}} + \ldots + a_s x^{\frac{1}{2}}}} = \lim_{x\to\infty} \frac{\Gamma(x+1)}{\sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[2s+1]{x^{\frac{2s+1}{2}}}} = \lim_{x\to\infty} \frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} = 1$$

2. RESULTS AND DISCUSSION

Formula (1) can be equivalently written as:

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[2s+1]{x^s\sqrt{x} + a_1 x^{s-1}\sqrt{x} + \ldots + a_s\sqrt{x}}$$

We introduce the following class of approximations:

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[s]{x\sqrt{x} + a\sqrt{x}} \quad (2)$$

depending on a real parameter *a*.

By using a method first introduced by Mortici in [4], we search for the value of *a* that provides the most accurate approximation (2). In this sense, we define the sequence w_n by the relations

$$\Gamma(n+1) = \sqrt{2\pi}\left(\frac{n}{e}\right)^n \sqrt[s]{n\sqrt{n} + a\sqrt{n}} \cdot \exp w_n, \ n = 1, 2, 3, \ldots$$

also called the relative error sequence. An approximation (2) will be considered more accurate as w_n goes faster to zero. As

$$w_n - w_{n+1} = n\ln\left(1+\frac{1}{n}\right) - 1 + \frac{1}{3}\ln\left(\left(n+1\right)^{\frac{3}{2}} + a\left(n+1\right)^{\frac{1}{2}}\right) - \frac{1}{3}\ln\left(n^{\frac{3}{2}} + an^{\frac{1}{2}}\right)$$

we use a computer software for symbolic computation to obtain

$$w_n - w_{n+1} = \left(\frac{1}{12} - \frac{1}{3}a\right)\frac{1}{n^2} + \left(\frac{1}{3}a + \frac{1}{3}a^2 - \frac{1}{12}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right) \qquad (3)$$

According to a result stated by Mortici [4], w_n converges as $n^{-(k-1)}$, when $w_n - w_{n+1}$ converges with a rate of convergence n^{-k}.

In consequence, the best estimate is obtained when the first coefficient in (3) vanishes, that is

$$\frac{1}{12} - \frac{1}{3}a = 0$$

The corresponding value $a = \frac{1}{4}$ produces the following approximation formula as $n \to \infty$:

$$\Gamma\left(x+1\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[3]{x\sqrt{x} + \frac{1}{4}\sqrt{x}}$$

This is much stronger than Stirling's formula, since:

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x < \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[3]{x\sqrt{x} + \frac{1}{4}\sqrt{x}} < \Gamma\left(x+1\right)$$

By using the same procedure, we deduced the following approximations of order five and seven:

$$\Gamma\left(x+1\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[5]{x^2\sqrt{x} + \frac{5}{12}x\sqrt{x} + \frac{5}{72}\sqrt{x}}$$

respective

$$\Gamma\left(x+1\right) \sim \tau\left(x\right) := \sqrt{2\pi}\left(\frac{x}{e}\right)^x \sqrt[7]{x^3\sqrt{x} + \frac{7}{12}x^2\sqrt{x} + \frac{49}{288}x\sqrt{x} + \frac{707}{51840}\sqrt{x}} \qquad (4)$$

The accuracy of such formulas increases as the root order is higher. Already our new formula (4) gives slightly better results than Ramanujan's formula, as we can see from the following comparison table.

x	$1-\frac{\Gamma\left(x+1\right)}{\rho\left(x\right)}$	$1-\frac{\Gamma\left(x+1\right)}{\tau\left(x\right)}$
50	1.4968×10^{-10}	1.4596×10^{-10}
100	9.4519×10^{-12}	9.2170×10^{-12}
250	2.4345×10^{-13}	2.3741×10^{-13}
1000	9.5389×10^{-16}	9.3021×10^{-16}

Let us remark that (4) can be equivalently written as:

$$\Gamma(x+1) \sim \tau(x) := \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{7}{12x}+\frac{49}{288x^2}+\frac{707}{51840x}\right)^{\frac{1}{7}}$$

Now we can see that this form allows us to establish similar formulas of type (4) of arbitrarily root order.

The method consists in transformation of the standard Stirling series of the logarithm of the gamma function

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\ldots\right)$$

which can be rewritten as:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left\{\exp\left(\frac{s}{12x}-\frac{s}{360x^3}+\frac{s}{1260x^5}-\frac{s}{1680x^7}+\ldots\right)\right\}^{\frac{1}{s}}$$

for every $s > 0$. Finally, the transition:

$$\exp\left(\frac{s}{12x}-\frac{s}{360x^3}+\frac{s}{1260x^5}-\frac{s}{1680x^7}+\ldots\right)=1+\frac{c_1}{x}+\frac{c_2}{x}+\ldots$$

can be performed using the following admissible transformation in asymptotic series theory:

$$\exp t = 1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots$$

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