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ASYMPTOTIC SERIES AND ESTIMATES OF A CONVERGENCE TO EULER-MASCHERONI CONSTANT

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Abstract. *We construct the asymptotic series and some estimates for a sequence converging to Euler-Mascheroni constant presented by Mortici in [3].*

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1. INTRODUCTION

The Euler-Mascheroni constant $\gamma = 0,577215...$, that was firstly studied by the Swiss matematician Leonhard Euler and Italian mathematician Lorenzo Mascheroni, has been defined as the limit of the sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n .$$

It is not yet known whether γ is a rational number or not; the inexistence of sufficient fast convergence to γ seems to be the key of the problem. In the recent past, many authors gave new fast convergences to γ.

We mention here the result of Mortici [3] who presented the sequences

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1} + \frac{1}{(6-2\sqrt{6})n} - \ln\left(n + \frac{1}{\sqrt{6}}\right)$$

and

$$v_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1} + \frac{1}{(6+2\sqrt{6})n} - \ln\left(n - \frac{1}{\sqrt{6}}\right)$$

showing that they converge to γ with the speed of convergence at n^{-3}. Mortici [3] introduced the following sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1} + \frac{1}{2n} - \frac{1}{2}\ln\left(n^2 - \frac{1}{6}\right) \quad (1)$$

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as the arithmetic mean of u_n and v_n and proved that t_n converges to the limit γ with the speed of convergence at n^{-4}. By using a method presented in [2], Mortici proved only that

$$\lim_{n\to\infty} n^4(t_n-\gamma)=\frac{11}{720}.$$

2. THE RESULTS

We prove the following estimates for t_n.

Theorem 1. For every integer $n\geq 1$, we have: $\frac{11}{720n^4}-\frac{29}{9072n^6}<t_n-\gamma<\frac{11}{720n^4}$.

Proof: Let us define the following sequences

$$p_n=(t_n-\gamma)-\left(\frac{11}{720n^4}-\frac{29}{9072n^6}\right) \quad \text{and} \quad q_n=(t_n-\gamma)-\frac{11}{720n^4}$$

that converge to zero. In order to prove that $p_n>0$ and $q_n<0$, it suffices to demonstrate that $(p_n)_{n\geq 1}$ is strictly decreasing and $(q_n)_{n\geq 1}$ is strictly increasing.

Let $p_{n+1}-p_n=f(n)$ and $q_{n+1}-q_n=g(n)$, where

$$f(x)=\frac{1}{2x+2}+\frac{1}{2x}-\frac{1}{2}\ln\left((x+1)^2-\frac{1}{6}\right)+\frac{1}{2}\ln\left(x^2-\frac{1}{6}\right)$$
$$-\left(\frac{11}{720(x+1)^4}-\frac{29}{9072(x+1)^6}\right)+\left(\frac{11}{720x^4}-\frac{29}{9072x^6}\right)$$

and

$$g(x)=\frac{1}{2x+2}+\frac{1}{2x}-\frac{1}{2}\ln\left((x+1)^2-\frac{1}{6}\right)+\frac{1}{2}\ln\left(x^2-\frac{1}{6}\right)-\frac{11}{720(x+1)^4}+\frac{11}{720x^4}.$$

We have

$$f'(x)=\frac{T(x)}{7560x^7(x+1)^7(12+6x^2+5)(6x^2-1)}$$

and

$$g'(x)=-\frac{S(x)}{180x^5(x+1)^5(12+6x^2+5)(6x^2-1)}$$

with

$$T(x)=1456645+8692767(x-1)+22115343(x-1)^2+31509417(x-1)^3$$
$$+27599465(x-1)^4+15254631(x-1)^5+5203555(x-1)^6+1002456(x-1)^7$$
$$+83538(x-1)^8$$

and

$$S(x)=3935+23631(x-1)+50006(x-1)^2+51576(x-1)^3+28171(x-1)^4$$
$$+7830(x-1)^5+870(x-1)^6.$$

Evidently, $f'>0$ on $(1,\infty)$ and $g'<0$ on $(1,\infty)$.It follows that f is strictly increasing on $(1,\infty)$ and g is strictly decreasing on $(1,\infty)$. Because $f(\infty)=g(\infty)=0$, we obtain $f<0$ on $(1,\infty)$ and $g>0$ on $(1,\infty)$.Thus, $(p_n)_{n\geq 1}$ is strictly decreasing and $(q_n)_{n\geq 1}$ is strictly increasing. As we explained, the conclusion follows. □

Now we construct the asymptotic series of the sequence t_n, using the representation of the harmonic sum

$$h_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}+\frac{1}{n}$$

in terms of digamma function

$$h_n=\gamma+\frac{1}{n}+\psi(n).$$

Here ψ is the digamma function defined by

$$\psi(x)=\frac{d}{dx}(\ln\Gamma(x))=\frac{\Gamma'(x)}{\Gamma(x)}.$$

See, e.g., [1]. The digamma function has the following asymptotic expansion

$$\psi(x)=\ln x-\frac{1}{2x}-\sum_{k=1}^{\infty}\frac{B_{2k}}{2kx^{2k}},$$

where B_j is the j th Bernoulli number given by

$$\frac{1}{e^t-1}+\frac{1}{2}-\frac{1}{t}=\sum_{j=1}^{\infty}(-1)^{j-1}\frac{t^{2j}}{(2j)!}B_j.$$

We are in a position to give the following

Theorem 2. The following asymptotic series is valid as $n\to\infty$:

$$t_n = \gamma + \sum_{k=2}^{\infty} \frac{1}{2k}\left\{\frac{1}{6^k} - B_{2k}\right\}\frac{1}{n^{2k}}.$$

Proof: We have

$$t_n = h_n - \frac{1}{2n} - \frac{1}{2}\ln\left(n^2 - \frac{1}{6}\right)$$

$$= \gamma + \frac{1}{n} + \psi(n) - \frac{1}{2n} - \frac{1}{2}\ln\left(n^2 - \frac{1}{6}\right)$$

$$= \gamma + \psi(n) - \ln n + \frac{1}{2n} - \frac{1}{2}\ln\left(1 - \frac{1}{6n^2}\right)$$

$$= \gamma - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kn^{2k}} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{k6^k n^{2k}}$$

$$= \gamma + \sum_{k=1}^{\infty} \frac{1}{2k}\left\{\frac{1}{6^k} - B_{2k}\right\}\frac{1}{n^{2k}}$$

and the conclusion follows. □

Explicitly, we have

$$t_n = \gamma + \frac{11}{720n^4} - \frac{29}{9072n^6} + \frac{221}{51840n^8} - \frac{6469}{855360n^{10}} + ...,$$

and by truncation of this series at any term, approximations of any accuracy n^{-2k} are obtained. Note that the first two terms for estimating t_n in Theorem 1 are the first terms of the previous asymptotic series.

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