

# LAPLACE DECOMPOSITION METHOD USING HE'S POLYNOMIAL TO BURGERS EQUATION

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*Manuscript received: 05.06.2014; Accepted paper: 18.06.2014;*

*Published online: 30.06.2014.*

**Abstract.** *In this paper, Laplace decomposition method using He's polynomial is employed to determine the exact solution of the Burger equation which 1-dimensional non-linear partial differential equation. This method is combined form of the Laplace Transforms Adomian decomposition method. The nonlinear term can be easily handled by the use of He's Polynomials. The explicit solutions obtained were compared the exact solutions. The method finds the solution without any restrictive assumptions and free from round-off errors and therefore reduce the numerical computation to a great extent. The method is tested on two examples and results show that new method is more effective and convenient to use and high accuracy of it is evident.*

**Keywords:** *Laplace transforms, Adomian Decomposition method, Burgers equation, Nonlinear partial differential equation.*

## 1. INTRODUCTION

Nonlinear phenomena play a crucial role in applied mathematics and physics. It is difficult to obtain the exact solution for these problems. Presently, an increasing interest of scientists and engineers has been devoted to the analytical techniques for solving nonlinear problems. In recent decades, there has been great development in the numerical analysis [1] and exact solution for nonlinear partial Differential Equations PDEs. There are many standard methods for solving Linear and nonlinear PDEs [2]; for instance, Backland transformation method [3], Variational iteration method [4-7], inverse scattering method [8], Hirota's bilinear method [9], homogeneous balance method [10] and He's homotopy perturbation method (HPM) [11-16], tanh-function method [17-18], sine – cosine method [19], and Exp-function method [20-22]. ADM was first proposed by G. Adomian [23]; unlike classical techniques, nonlinear equations are solved easily and more accurately via ADM.

Laplace Adomian's Decomposition Method (LADM) was first introduced by Suheil A. Khuri [24-25], and has been successfully used to find the solution of differential equations [26-31]. This Method has been applied successfully to find the exact solution of the Bratu and Duffing equation in [27]. The Significant advantage of this method is its capability of combining the two powerful methods to obtain exact solution for non-linear

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equations. The basic motive of the present study is the implementation of the reliable modifications of Adomian decomposition method to the different variants to Burger's equation.

## 2. 2. LAPLACE DECOMPOSITION METHOD

To illustrate the basic idea of this method, we consider a general nonlinear non-homogeneous partial differential equation with initial conditions of the form

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \quad (1)$$

$$u(x,0) = h(x), \quad u_t(x,0) = f(x) \quad (2)$$

where  $D$  is the second order linear differential operator  $D = \frac{\partial^2}{\partial t^2}$ ,  $R$  is the linear differential operator of less order than  $D$ ,  $N$  represent the general nonlinear differential operator and  $g(x,t)$  is the source term. Taking Laplace Transform on both sides of Eq. (1)

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)] \quad (3)$$

$$Lu(x,t) = \frac{h(x)}{s} + \frac{f(x)}{s^2} - \frac{L}{s^2}[Ru(x,t)] + \frac{L}{s^2}[g(x,t)] - \frac{L}{s^2}[Nu(x,t)] \quad (4)$$

Operating with the inverse Laplace transform on both sides of Eq. (4), we get

$$u(x,t) = A(x,t) - L^{-1} \left[ \frac{L}{s^2} [Ru(x,t) + Nu(x,t)] \right] \quad (5)$$

where  $A(x,t)$  represent the term arising from the source term and the prescribed initial conditions. Assume the solution of Eq. (1) to be in the form

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots \quad (6)$$

The nonlinear operator is decomposed as

$$Nu(x,t) = \sum_{j=0}^{\infty} p^j H_j \quad (7)$$

where He's polynomial given by

$$H_n(u_0, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left( N \left( \sum_{i=0}^{\infty} p^i u_i \right) \right)_{p=0}, \quad n = 0, 1, 2, \dots$$

substituting Eq. (6) & (7) in Eq. (5), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = A(x,t) - p \left\{ L^{-1} \left[ \frac{L}{s^2} \left[ R \sum_{n=0}^{\infty} p^n H_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u) \right] \right] \right\} \quad (8)$$

Comparing the coefficient of like powers of  $p$ , the following approximations are obtained [4, 7]

$$\begin{aligned} p^0 : u_0(x,t) &= -\frac{L}{s^2} [Ru_0(x,t) + H_0(u)] \\ p^1 : u_1(x,t) &= -\frac{L}{s^2} [Ru_1(x,t) + H_1(u)] \\ p^2 : u_2(x,t) &= -\frac{L}{s^2} [Ru_2(x,t) + H_2(u)] \\ p^3 : u_3(x,t) &= -\frac{L}{s^2} [Ru_3(x,t) + H_3(u)] \end{aligned}$$

The best approximations for the solutions are

$$u = \lim_{p \rightarrow 1} u_n = u_0 + u_1 + u_2 + \dots \quad (9)$$

### 3. NUMERICAL EXAMPLES

In this section, the ADM using He's polynomials is implemented for tackling Burger's equations with initial conditions. We demonstrate the effectiveness of this method with two examples. Numerical results obtained by the proposed method are compared with known results.

**Example 3.1.** Consider one-dimensional burger's equation of the form

$$u_t = u_{xx} - uu_x \quad (10)$$

The initial condition is

$$u(x,0) = 1 - \frac{2}{x} \quad (11)$$

Taking Laplace transform on both sides

$$L[u_t] = L[u_{xx}] - L[u u_x] \quad (12)$$

This can be written as

$$[su(x, s) - u(x, 0)] = L\left[\frac{\partial^2 u}{\partial x^2}\right] - L\left[u \frac{\partial u}{\partial x}\right] \quad (13)$$

On applying the above specified initial condition we get

$$su(x, s) - \left(1 - \frac{2}{x}\right) = L\left[\frac{\partial^2 u}{\partial x^2}\right] - L\left[u \frac{\partial u}{\partial x}\right] \quad (14)$$

$$u(x, s) = \frac{1}{s} - \frac{2}{sx} + L\left[\frac{\partial^2 u}{\partial x^2}\right] - \frac{L}{s}\left[u \frac{\partial u}{\partial x}\right] \quad (15)$$

applying the inverse Laplace transform on both sides of Eq. (15), we get

$$\begin{aligned} L^{-1}[u(x, s)] &= L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{2}{sx}\right] + L^{-1}\left[\frac{L}{s}\left[\frac{\partial^2 u}{\partial x^2}\right]\right] - L^{-1}\left[\frac{L}{s}\left[u \frac{\partial u}{\partial x}\right]\right] \\ u(x, t) &= 1 - \frac{2}{x} + L^{-1}\left[\frac{L}{s}\left[\frac{\partial^2 u}{\partial x^2}\right]\right] - L^{-1}\left[\frac{L}{s}\left[u \frac{\partial u}{\partial x}\right]\right] \end{aligned} \quad (16)$$

we decompose the solution as an infinite sum given below

$$u(x) = \sum_{i=0}^{\infty} p^i u_i \quad (17)$$

Using Eq. (17) into Eq. (16), the recursive relation is given below

$$\begin{aligned} u_0 + pu_1 + p^2u_2 + \dots &= \left(1 - \frac{2}{x}\right) + pL^{-1}\left[\frac{L}{s}\left[\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_3}{\partial x^2} + \dots\right)\right]\right] - \\ &pL^{-1}\left[\frac{L}{s}\left[\left(u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots\right)\left(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} + \dots\right)\right]\right] \end{aligned}$$

Comparing the coefficient of various power of  $p$ , we get

$$\begin{aligned}
p^0 : u_0(x, t) &= 1 - \frac{2}{x} \\
p^1 : u_1(x, t) &= L^{-1} \left[ \frac{L \left[ \frac{\partial^2 u_0}{\partial x^2} \right]}{s} \right] - L^{-1} \left[ \frac{L \left[ u_0 \frac{\partial u_0}{\partial x} \right]}{s} \right] = -\frac{2}{x^2} t \\
p^2 : u_2(x, t) &= L^{-1} \left[ \frac{L \left[ \frac{\partial^2 u_1}{\partial x^2} \right]}{s} \right] - L^{-1} \left[ \frac{L \left[ u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x} \right]}{s} \right] = -\frac{2}{x^3} t^2 \\
p^3 : u_3(x, t) &= L^{-1} \left[ \frac{L \left[ \frac{\partial^2 u_2}{\partial x^2} \right]}{s} \right] - L^{-1} \left[ \frac{L \left[ u_2 \frac{\partial u_0}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial u_2}{\partial x} \right]}{s} \right] = -\frac{2}{x^4} t^3 \\
p^4 : u_4(x, t) &= L^{-1} \left[ \frac{L \left[ \frac{\partial^2 u_3}{\partial x^2} \right]}{s} \right] - L^{-1} \left[ \frac{L \left[ u_3 \frac{\partial u_0}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_2}{\partial x} + u_0 \frac{\partial u_3}{\partial x} \right]}{s} \right] = -\frac{2}{x^5} t^4 \\
p^5 : u_5(x, t) &= L^{-1} \left[ \frac{L \left[ \frac{\partial^2 u_4}{\partial x^2} \right]}{s} \right] - L^{-1} \left[ \frac{L \left[ u_4 \frac{\partial u_0}{\partial x} + u_3 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_3}{\partial x} + u_0 \frac{\partial u_4}{\partial x} \right]}{s} \right] = -\frac{2}{x^6} t^5 \\
&\vdots
\end{aligned}$$

Then the solution  $u(x, t)$  in series form is given by

$$u(x, t) = 1 - \frac{2}{x} + \left( -\frac{2}{x^2} t \right) + \left( -\frac{2}{x^3} t^2 \right) + \left( -\frac{2}{x^4} t^3 \right) + \left( -\frac{2}{x^5} t^4 \right) + \dots$$

and in closed form is

$$u(x, t) = 1 - \frac{2}{(x-t)} \quad (18)$$

**Example 3.2.** Consider one-dimensional burger's equation of the form

$$u_t = u_{xx} - uu_x \quad (19)$$

Subject to initial condition

$$u(x, 0) = x \quad (20)$$

Taking Laplace transform on both sides

$$L[u_t] = L[u_{xx}] - L[u u_x] \quad (21)$$

This can be written as

$$\left[ su(x, s) - u(x, 0) \right] = L \left[ \frac{\partial^2 u}{\partial x^2} \right] - L \left[ u \frac{\partial u}{\partial x} \right] \quad (22)$$

On applying the above specified initial condition we get

$$[su(x, s) - x] = L\left[\frac{\partial^2 u}{\partial x^2}\right] - L\left[u \frac{\partial u}{\partial x}\right] \quad (23)$$

$$u(x, s) = \frac{x}{s} + \frac{L}{s}\left[\frac{\partial^2 u}{\partial x^2}\right] - \frac{L}{s}\left[u \frac{\partial u}{\partial x}\right] \quad (24)$$

applying the inverse Laplace transform to both sides of Eq. (24), we get

$$L^{-1}[u(x, s)] = L^{-1}\left[\frac{x}{s}\right] + L^{-1}\left[\frac{L}{s}\left[\frac{\partial^2 u}{\partial x^2}\right]\right] - L^{-1}\left[\frac{L}{s}\left[u \frac{\partial u}{\partial x}\right]\right]$$

$$u(x, t) = x + L^{-1}\left[\frac{L}{s}\left[\frac{\partial^2 u}{\partial x^2}\right]\right] - L^{-1}\left[\frac{L}{s}\left[u \frac{\partial u}{\partial x}\right]\right] \quad (25)$$

we decompose the solution as an infinite sum given below

$$u(x) = \sum_{i=0}^{\infty} p^i u_i \quad (26)$$

Using Eq. (26) into Eq. (25), the recursive relation is given below

$$\sum_{n=0}^{\infty} p^n u_n(x, t) = x + p L^{-1}\left[\frac{L}{s}\left[\left(\sum_{n=0}^{\infty} p^n u_n(x, t)\right)_{xx}\right]\right]$$

$$- p L^{-1}\left[\frac{L}{s}\left[\left(\sum_{n=0}^{\infty} p^n u_n(x, t)\right)\left(\sum_{n=0}^{\infty} p^n u_n(x, t)\right)_x\right]\right] \quad (27)$$

$$u_0 + pu_1 + p^2u_2 + \dots = x + pL^{-1}\left[\frac{L}{s}\left[\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_3}{\partial x^2} + \dots\right)\right]\right]$$

$$- pL^{-1}\left[\frac{L}{s}\left[\left(u_0 + pu_1 + p^2u_2 + p^3u_3 \dots\right)\left(\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} + \frac{\partial u_3}{\partial x} + \dots\right)\right]\right]$$

Consequently, we have

$$p^0 : u_0(x, t) = x$$

$$p^1 : u_1(x, t) = L^{-1} \left[ \frac{L}{s} \left[ \frac{\partial^2 u_0}{\partial x^2} \right] \right] - L^{-1} \left[ \frac{L}{s} \left[ u_0 \frac{\partial u_0}{\partial x} \right] \right] = -xt$$

$$p^2 : u_2(x, t) = L^{-1} \left[ \frac{L}{s} \left[ \frac{\partial^2 u_1}{\partial x^2} \right] \right] - L^{-1} \left[ \frac{L}{s} \left[ u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x} \right] \right] = \frac{2}{2!} xt^2$$

$$p^3 : u_3(x, t) = L^{-1} \left[ \frac{L}{s} \left[ \frac{\partial^2 u_2}{\partial x^2} \right] \right] - L^{-1} \left[ \frac{L}{s} \left[ u_2 \frac{\partial u_0}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial u_2}{\partial x} \right] \right] = -\frac{4}{3!} xt^3$$

$$p^4 : u_4(x, t) = L^{-1} \left[ \frac{L}{s} \left[ \frac{\partial^2 u_3}{\partial x^2} \right] \right] - L^{-1} \left[ \frac{L}{s} \left[ u_3 \frac{\partial u_0}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_2}{\partial x} + u_0 \frac{\partial u_3}{\partial x} \right] \right] = \frac{10}{4!} t^4$$

$$p^5 : u_5(x, t) = L^{-1} \left[ \frac{L}{s} \left[ \frac{\partial^2 u_4}{\partial x^2} \right] \right] - L^{-1} \left[ \frac{L}{s} \left[ u_4 \frac{\partial u_0}{\partial x} + u_3 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_3}{\partial x} + u_0 \frac{\partial u_4}{\partial x} \right] \right] = \frac{92}{5!} xt^5$$

and so on.

Then the solution  $u(x, t)$  in series form is given by

$$u(x, t) = x - xt + \frac{2}{2!} xt^2 - \frac{4}{3!} xt^3 + \frac{10}{4!} xt^4 + \frac{92}{5!} xt^5 + \dots \quad (28)$$

#### 4. CONCLUSIONS

In this paper, we have successfully developed Laplace decomposition method (LDM) using He's polynomials for solution of Burger's Equation. The method is applied in a direct way without any linearization or discretization. From the obtained results it may concluded that LDM is an extremely powerful and efficient method in finding analytical solutions for a number of nonlinear problems and can be used in finding the exact solutions for wide classes of problems.

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