ORIGINAL PAPER

NETWORKS DEFINED BY ALTERNATIVE RECURRENCES

VASILE POP[1]

Manuscript received: 25.05.2014; Accepted paper: 10.06.2014;
Published online: 30.06.2014.

Abstract. *If F is a family of functions, we define a sequence by alternative recurrence with respect to the family F a sequence* $(a_n)_n$ *that verifies the relation* $a_{n+1} \in \{ f(a_n, a_{n-1}) \mid f \in F\}$ *for any* $n \geq 1$, *where the first two terms* a_0, a_1 *are given. We prove that for particular cases of family we can obtain the regular networks in the complex plane.*

Keywords: *alternative recurrence, regular networks, complex plane.*

1. SEQUENCES DEFINED BY ALTERNATIVE RECURRENCES

Let $F = \{ f_i \mid i \in I\}$ be a family of functions $f_i : \mathbf{C} \to \mathbf{C}$ or $f_i : \mathbf{R} \to \mathbf{R}$, $i \in I$, where I is a set of indices.

Definition 1.1. A *sequence* defined by an alternative recurrence of order I determined by F is a family of sequences $(a_n)_{n\in\mathbf{N}}$ which verifies the recurrence relation $a_{n+1} \in \{ f_i(a_n) \mid i \in I\}$, for any $n \in \mathbf{N}$, where the first term $a_0 \in \mathbf{C}$ or $a_0 \in \mathbf{R}$ is given.

We denote by $A(F, a_0)$ the set of all these sequences and by $M(F, a_0)$ the set of all terms of all these sequences (the set of all numbers z for which there is a sequence $(a_n)_n \in A(F, a_0)$ and $k \in \mathbf{N}$ such that $a_k = z$).

Remark 1.1. a) In the set $A(F, a_0)$ there are also sequences defined by ordinary recurrences of order I. If we fix an indice $i_0 \in I$ then such a sequence is $(a_n)_n$ where $a_{n+1} = f_{i_0}(a_n)$, for any $n \in \mathbf{N}$. The general term of such a sequence is $a_n = f_{i_0}^n(a_0)$, where $f_{i_0}^n = f_{i_0} \circ f_{i_0} \circ \ldots \circ f_{i_0}$.

b) If $F = \{ f_1, f_2\}$ and $f_1 \circ f_2 = f_2 \circ f_1$, then

$$M(F, a_0) = \{ f_1^n (f_2^n (a_0)) \mid m, n \in \mathbf{N}\}.$$

[1] Technical University of Cluj-Napoca, Department of Mathematics, 400114 Cluj-Napoca, Romania.
E-mail: Vasile.Pop@math.utcluj.ro.

c) The set $A(F,a_0)$ may contain convergent and divergent sequences, also bounded or unbounded sequences, monotone or non-monotone sequences, as it can be seen from the next example.

Example 1.1. Let $F=\{f_1,f_2,f_3\}$, $f_1,f_2,f_3:\mathbf{R}\to\mathbf{R}$, $f_1=1$, $f_2=-1$, $f_3(x)=e^x-1$, $x\in\mathbf{R}$ and the sequences $(a_n^{(1)})_n$, $(a_n^{(2)})_n$, $(a_n^{(3)})_n$ defined by alternative recurrence:

$$a_{n+1}^{(1)}=\begin{cases} f_1(a_n), & n \text{ is even} \\ f_2(a_n), & n \text{ is odd}\end{cases},\ a_n^{(1)}=(-1)^{n+1}, \text{ for } n\geq 1$$

which is bounded, non-monotone and divergent.

$$a_{n+1}^{(2)}=\begin{cases} f_2(a_n), & n=0 \\ f_3(a_n), & n\geq 1\end{cases}$$

which is monotone, convergent and $\lim_{n\to\infty} a_n^{(2)}=0$

$$a_{n+1}^{(3)}=\begin{cases} f_1(a_n), & n=0 \\ f_3(a_n), & n\geq 1\end{cases}$$

which is monotone, unbounded and $\lim_{n\to\infty} a_n^{(3)}=\infty$.

Remark 1.2. An open problem related to the study of alternative recurrence sequences is the following one:

Open problem. Find the conditions that should be satisfied by functions in *F* and the initial term a_0 such that all sequences in $A(F,a_0)$ have the same behavior related to monotonicity, boundedness and convergence. An example can be found in [6].

Let $G=\{g_j \mid j\in J\}$ be a family of functions of two variables $g_j:\mathbf{C}\times\mathbf{C}\to\mathbf{C}$ or $g_j:\mathbf{R}\times\mathbf{R}\to\mathbf{R}$.

Definition 1.2. We call a *sequence* defined by an alternative recurrence of order II, determined by the family of maps *G*, any sequence $(a_n)_n$ which verifies the recurrence relation $a_{n+1}\in\{g_j(a_n,a_{n-1}) \mid j\in J\}$ for any $n\geq 1$, where the first two terms $a_0,a_1\in\mathbf{C}$ or $a_0,a_1\in\mathbf{R}$ are given (the initial conditions).

We denote by $A(G,a_0,a_1)$ the set of all these sequences and with $M(G,a_0,a_1)$ the set of all terms of these sequences.

Remark 1.3. If we consider a family of maps of *k* variables $H=\{h_i \mid i\in I\}$, with $h_i:\mathbf{C}^k\to\mathbf{C}$ or $h_i:\mathbf{R}^k\to\mathbf{R}$, similarly we can define sequences determined by alternative recurrences of order *k*, in which each term a_n has the alternative of being chosen by selecting maps $h_{i(n)}\in H$ by the formula $a_n=h_{i(n)}(a_{n-1},a_{n-2},...,a_{n-k})$.

Similarly we have open problems regarding the simultaneous behavior of the sequences from $A(G,a_0,a_1)$, to the form or the structured of the set $M(G,a_0,a_1)$ or the density of $M(G,a_0,a_1)$ in **C** or **R**.

2. PLANE POLYGONAL NETWORKS

Definition 2.1. We call *plane polygonal network* any cover of the plane $(\pi)=\bigcup_{i\in I} P_i$, by polygons P_i, $i\in I$ in which any two polygons have disjoint interiors and any two polygons with common points (on this boundaries) have exactly one common vertex or exactly one common edge. The polygonal edges are called the *network segments* and the polygonal vertices are called the *network nodes* [4, 5].

Remark 2.1. Generally we consider polygonal networks in which all the polygons from the cover are congruent.

Recall the following classic result [2, 3].

Theorem 2.1. There are only three type of plane polygonal network formed by congruent regular polygons: the triangular network, the square network (plane lattice) and the hexagonal network.

Remark 2.2. a) The set of the nodes of the triangular network of edge 1 in the complex plane is

$$N_3=\{m+n\varepsilon \mid m,n\in \mathbf{Z}\}\text{, where } \varepsilon=\frac{1+i\sqrt{3}}{2}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}.$$

b) The set of nodes of the square network of edge 1 in the complex plane is

$$N_4=\{m+ni \mid m,n\in \mathbf{Z}\}=\mathbf{Z}[i].$$

c) The set of nodes of the hexagonal network of edge 1 is

$$N_6=\left\{3k+p\sqrt{3}\,i, 3k+1+p\sqrt{3}\,i, 3k+1+\frac{1}{2}+\left(p+\frac{1}{2}\right)\sqrt{3}\,i+3k+2+\frac{1}{2}+\left(p+\frac{1}{2}\right)\sqrt{3}\,i \mid k,p\in\mathbf{Z}\right\}$$

3. REGULAR NETWORKS DEFINED BY ALTERNATIVE RECURRENCES

We will show that any of the three types of regular polygonal networks from Theorem 2.1 can be obtained using alternative recurrences with two linear maps of the form

$$f_a(z,u)=(1-a)z+a\cdot u\ ,\ z,u\in\mathbf{C}$$

and

$$f_{\overline{a}}(z,u)=(1-\overline{a})z+\overline{a}\cdot u\ ,\ z,u\in\mathbf{C},$$

where $a \in \mathbf{C}$ is specially chosen.

Let $a \in \mathbf{C}$ be fixed and $F = \{f_a, f_{\overline{a}}\}$.

We consider the sequences defined by alternative recurrence:

$$z_{n+1} \in \{(1-a)z_n + a \cdot z_{n-1}, (1-\overline{a})z_n + \overline{a} \cdot z_{n-1}\},\ n \geq 1$$

and let $A(F, z_0, z_1)$ be the set of all these sequences, where $z_0, z_1 \in \mathbf{C}$ are fixed numbers with $z_0 \neq z_1$.

Theorem 3.1. If $a \in \mathbf{C}$ and $|a| < 1$ then all the sequences $(z_n)_n$ which verifies the recurrence:

$$z_{n+1} \in \{(1-a)z_n + a \cdot z_{n-1}, (1-\overline{a})z_n + \overline{a} \cdot z_{n-1}\} \text{ for any } n \geq 1,\ z_0, z_1 \in \mathbf{C}$$

are convergent.

Proof: We have

$$z_{n+1} - z_n = -a(z_n - z_{n-1})$$

or

$$z_{n-1} - z_n = -\overline{a}(z_n - z_{n-1}).$$

In both cases

$$|z_{n+1} - z_n| = |a| \cdot |z_n - z_{n-1}| = \ldots = |a|^n \cdot |z_1 - z_0|.$$

We have:

$$\begin{aligned} |z_{n+p} - z_n| &\leq |z_{n+p} - z_{n+p-1}| + |z_{n+p-1} - z_{n+p-2}| + \ldots + |z_{n+1} - z_n| \\ &\leq (|a|^{n+p-1} + |a|^{n+p-2} + \ldots + |a|^n)|z_1 - z_0| \\ &= |a|^n \cdot \frac{1-|a|^p}{1-|a|} \cdot |z_1 - z_0| < \frac{|z_1 - z_0|}{1-|a|} \cdot |a|^n, \end{aligned}$$

so for $|a| < 1$ the sequence $(z_n)_n$ is a Cauchy sequence, hence convergent.

Theorem 3.2. The sequence $(z_n)_n$ defined by the recurrence relation:

$$z_{n+1} = (1-a)z_n + a \cdot z_{n-1},\quad n \geq 1,\quad z_0, z_1 \in \mathbf{C},\quad z_0 \neq z_1$$

is a periodic sequence if and only if there is $k \in \mathbf{N}$, $k \geq 2$ such that $a^k = 1$ and $a \neq -1$.

Proof: We have:

$$z_{n+1} - z_n = (-a)(z_n - z_{n-1}) = \ldots = (-a)^n (z_1 - z_0)$$

and then

$$\begin{aligned} z_n &= (z_n - z_{n-1}) + (z_{n-1} - z_{n-2}) + \ldots + (z_1 - z_0) + z_0 \\ &= z_0 + (z_1 - z_0)(1 - a + a^2 - \ldots + (-a)^{n-1}) \\ &= z_0 + (z_1 - z_0)\frac{1-(-a)^n}{1+a}, \end{aligned}$$

for $a \neq -1$.

Being defined by a recurrence of order II, the sequence is periodic if and only if there is $n \geq 1$ such that $z_n = z_0$ and $z_{n+1} = z_1 \Leftrightarrow 1-(-a)^n = 0 \Leftrightarrow (-a)^n = 1 \Leftrightarrow a^{2n} = 1$.

Corollary 3.2. a) For $a = \frac{1 \pm i\sqrt{3}}{2}$ the sequence $(z_n)_n$ is periodic of period 3.

b) For $a = \pm i$ the sequence $(z_n)_n$ is periodic of period 4.

c) For $a = \frac{-1 \pm i\sqrt{3}}{2}$ the sequence $(z_n)_n$ is periodic of period 6.

Theorem 3.3. For $\varepsilon = \frac{1 \pm i\sqrt{3}}{2}$ the set of terms of all sequences $(z_n)_n$ which verifies the recurrence:

$$z_{n+1} \in \{(1-\varepsilon)z_n + \varepsilon \cdot z_{n-1}, (1-\bar{\varepsilon})z_n + \bar{\varepsilon} \cdot z_{n-1}\} \text{ for any } n \geq 1,\ z_0 = 0,\ z_1 = 1$$

is the set of nodes of triangular network of edge 1, i.e. $A(F,0,1) = N_3$.

Proof: From $z_{n+1} - z_n = \varepsilon(z_{n-1} - z_n)$ or $z_{n+1} - z_n = \bar{\varepsilon}(z_{n-1} - z_n)$ it follows that

$$| z_{n+1} - z_n | = | z_n - z_{n-1} | = \ldots = | z_1 - z_0 | = 1$$

and the angle $\angle A_{n-1}A_nA_{n+1}$ is $\frac{\pi}{3}$. We can see the sequence $(z_n)_n$ as a path $A_0, A_1, \ldots, A_n, \ldots$ with nodes in the triangular network which begin in $A_0 = 0$ and the first step is A_0A_1, $A_1 = 1$.

To show that we can find any node of the network through a path is enough to show that for any point 0 of network we can attend in any neighbour point (at a distance 1 from it).

Let $ABCDEF$ be a hexagon with edge 1 and center 0, its vertices being nodes of triangular network and suppose that the step which we reach the node 0 is AO. The path $AOBCODEOF$ is passing through all neighbouring points.

Theorem 3.4. The set of all terms of sequences $(z_n)_n$ which verifies the alternative recurrences:

$$z_{n+1} \in \{(1-i)z_n + iz_{n-1}, (1+i)z_n - iz_{n-1}\} \text{ for any } n \geq 1,\ z_0 = 0,\ z_1 = 1$$

is the lattice network $\mathbf{Z}[i]$.

Proof: Similarly to Theorem 3.3 we have:

$$|z_{n+1}-z_n|=|z_n-z_{n-1}|=...=|z_1-z_0|=1$$

and the angle $\angle A_{n-1}A_nA_{n+1}=\dfrac{\pi}{2}$. We regard the sequence $(z_n)_n$ as a path $A_0, A_1,..., A_n,...$ in the lattice network which starts at $A_0=0$, $A_1=1$.

It is enough to show that from any node 0 we can reach to any of its neighbours: A, B, C, D. If the step through which we reach 0 is AO, then to B we can reach directly through A, O, B, in D we reach directly through A, O, D and in C we can reach by A, O, B, E, C where E is a vertex of square $COBE$.

Theorem 3.5. For $\omega=\dfrac{-1+i\sqrt{3}}{2}$ the set of all terms of all sequences given by the alternative recurrence:

$$z_{n+1}\in\{(1-\omega)z_n+\omega\cdot z_{n-1},(1-\overline{\omega})z_n+\overline{\omega}\cdot z_{n-1}\} \text{ for any } n\geq 1,\ z_0=0,\ z_1=1$$

is the set of nodes of regular hexagonal network of edge 1: $M(F,0,1)=N_6$.

Proof: As in the preceding theorem it is enough to show that from any node 0 of hexagonal network we can reach in any of the three neighbour nodes A, B, C. From the relation $z_{n+1}-z_n=\omega(z_{n-1}-z_n)$ or $z_{n+1}-z_n=\overline{\omega}(z_{n-1}-z_n)$ it follows that any step has length 1 and the angle $\angle A_{n-1}A_nA_{n+1}$ is $\dfrac{2\pi}{3}$. If in 0 we reach from A then in B and C we can reach directly: $\angle AOB=\angle AOC=\dfrac{2\pi}{3}$.

Remark 3.1. For the case of Theorem 3.3, 3.4, 3.5 we can also ask the following combinatorial problems:

a) for what values of n there are sequences with $z_n=0$?

b) for $k\in\mathbf{N}$, how many sequences $(z_n)_n$ have the property that $z_k=0$?

c) for what values of a from Theorem 3.1 the set of all terms of sequences $(z_n)_n$ is dense in **C**?

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