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UNIFIED FRACTIONAL INTEGRAL FORMULAE FOR THE GENERALIZED MITTAG-LEFFLER FUNCTIONS

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Abstract. *The fractional calculus operators has gained noticeable importance and popularity due to its established applications in describing/modeling and solving various integral equations, ordinary differential equations and partial differential equations in many fields of science and engineering. The Mittag-Leffler functions are important special functions, that provides solutions to number of problems formulated in terms of fractional order differential, integral and difference equations; therefore, it has recently become a subject of interest for many authors in the field of fractional calculus and its applications. The aim of this paper is to evaluate two unified fractional integrals involving the product of generalized Mittag-Leffler function and Appell function* $F_3(.)$. *These integrals are further applied in proving two theorems on Marichev-Saigo-Maeda fractional integral operators. The results are expressed in terms of generalized Wright function and hypergeometric functions* ${}_pF_q(.)$. *Further, we point out also their relevance.*

Keywords: *Marichev-Saigo-Maeda fractional integral operators, generalized Mittag-Leffler function, generalized Wright function, generalized hypergeometric series.*

1. INTRODUCTION

The fractional calculus operators involving various special functions have found significant importance and applications in modeling of relevant systems in various fields science and engineering, such as turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics, and in quantum mechanics. Therefore, a remarkably large number of authors have studied, in depth, the properties, applications, and different extensions of various operators of fractional calculus. For detailed account of fractional calculus operators along with their properties and applications, one may refer to the research monographs by Miller and Ross [4], Samko *et al.* [14] and Kiryakova [2].

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In 1903, Gosta Mittag-Leffler [5] introduced the function $E_\alpha(z)$, defined by:

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+1)}z^n \quad (\alpha\in C);\ \mathrm{Re}(\alpha)>0 \tag{1.1}$$

A further, two-index generalization of this function was given by Wiman [17] as:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}z^n \quad (\alpha,\beta\in C), \tag{1.2}$$

where $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$.

By means of the series representation a generalization of Mittag-Leffler function (1.2) is introduced by Prabhakar [7] as:

$$E_{\beta,\gamma}^{\delta}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_n}{\Gamma(\beta n+\gamma)n!}z^n \tag{1.3}$$

where $\beta,\gamma,\delta\in C$ $(\mathrm{Re}(\beta)>0)$. Further, it is an entire function of order $[\mathrm{Re}(\beta)]^{-1}$ [7].

Since the Mittag-Leffler function provides solutions to certain problems formulated in terms of fractional order differential, integral and difference equations, therefore, a number of useful generalization of the this function has been introduced and studied many authors. Recently, Salim and Faraj [13] has introduced and studied a new generalization of the Mittag-Leffler function, by means of the power series:

$$E_{\nu,\rho,p}^{\delta,\xi,q}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_{qn}}{\Gamma(\nu n+\rho)(\xi)_{pn}}z^n \tag{1.4}$$

where $\nu,\rho,\delta,\xi\in C;\ \Re(\nu),\Re(\rho),\Re(\delta),\Re(\xi),p,q>0$ such that $q\le\Re(\nu)+p$.

The generalized Wright hypergeometric function ${}_p\psi_q(z)$, for $z\in C$, complex $a_i,b_j\in C$, and $\alpha_i,\beta_j\in R$ $(\alpha_i,\beta_j\neq 0;\ i=1,2,...,p;\ j=1,2,...,q)$ is defined as below:

$${}_p\psi_q(z)={}_p\psi_q\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\Big|z\right]=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)z^k}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)k!}. \tag{1.5}$$

Wright [18] introduced the generalized Wright function (1.5) and proved several theorems on the asymptotic expansion of ${}_p\psi_q(z)$ [18-20] for all values of the argument z, under the condition:

$$\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>-1.$$

The generalized hypergeometric function for complex $a_i, b_j \in \mathbb{C}$ and $b_j \neq 0, -1, \ldots\ (i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, q)$ is given by the power series [1]:

$$ {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \sum_{r=0}^{\infty} \frac{(a_1)_r \cdots (a_p)_r z^r}{(b_1)_r \cdots (b_q)_r r!}, \tag{1.6} $$

where for convergence, we have $|z| < 1$ if $p = q + 1$ and for any z if $p \leq q$. The function (1.6) is a special case of the generalized Wright function (1.5) for $\alpha_1 = \ldots = \alpha_p = \beta_1 = \ldots = \beta_q = 1$:

$$ {}_pF_q(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{i=1}^{p} \Gamma(a_i)} \, {}_p\psi_q \left[\begin{matrix} (a_i, 1)_{1,\ p} \\ (b_j, 1)_{1,\ q} \end{matrix} \middle| z \right] \tag{1.7} $$

A useful generalization of the hypergeometric fractional integrals, including the Saigo operators [10-11], has been introduced by Marichev [3] (see details in Samko *et al.* [14]) and later extended and studied by Saigo and Maeda [12] in term of any complex order with Appell function $F_3(.)$ in the kernel, as follows:

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ and $x > 0$, then the generalized fractional calculus operators (the Marichev-Saigo-Maeda operators) involving the Appell function, or Horn's F_3-function are defined by the following equations:

$$ \left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t)dt \quad (\Re(\gamma) > 0 \tag{1.8} $$

$$ \left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^{\infty} (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{x}{t}, 1 - \frac{t}{x}\right) f(t)dt \ \ (\Re(\gamma) > 0 \tag{1.9} $$

For the definition of the Appell function $F_3(.)$ the interested reader may refer to the monograph by Srivastava and Karlson [16] (see [1, 6]). Following Saigo *et al.* [10, 15], the image formulas for a power function, under operators (1.8) and (1.9), are given by:

$$ \left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}\right)(x) = \Gamma\left[\begin{matrix} \rho, \rho+\gamma-\alpha-\alpha'-\beta, \rho+\beta'-\alpha' \\ \rho+\beta', \rho+\gamma-\alpha-\alpha', \rho+\gamma-\alpha'-\beta \end{matrix} \right] x^{\rho-\alpha-\alpha'+\gamma-1} \tag{1.10} $$

where $\Re(\rho) > max\{0\ \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\}$ and $\Re(\gamma) > 0$.

$$ \left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1} \Gamma\left[\begin{matrix} 1-\rho-\beta, 1-\rho-\gamma+\alpha+\alpha',\ 1-\rho+\alpha+\beta'-\gamma \\ 1-\rho, 1-\rho+\alpha+\alpha'+\beta'-\gamma,\ 1-\rho+\alpha-\beta \end{matrix} \right] \tag{1.11} $$

wher $\Re(\rho) < 1 + min\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\}$ and $\Re(\gamma) > 0$.

The symbol occurring in (1.10) and (1.11) is given by:

$$ \Gamma\left[\begin{matrix} a, b, c \\ d, e, f \end{matrix} \right] = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}. $$

The computations of fractional integrals (and fractional derivatives) of special functions of one and more variables are important from the point of view of the usefulness of these results in the evaluation of generalized integrals and the solution of differential and integral equations (for example see [8]-[9]). Motivated by these avenues of applications, here we establish two image formulas for the generalized Mittag-Leffler function (1.4), involving left and right sided operators of Saigo-Meada fractional integral operators [12], in term of the generalized Wright function [18].

2. MAIN RESULTS

In this section, we establish image formulas for the generalized Mittag-Leffler function involving left and right sided operators of Saigo-Meada fractional integral operators (1.8) and (1.9), in term of the generalized Wright function. These formulas are given by the following theorems:

Theorem 2.1. Let $\alpha,\alpha',\beta,\beta',\gamma,\upsilon,\rho\in\mathbb{C}$ and $p,q>0$, $\nu>0$, $q\leq\Re(\nu)+p$ be such that

$$\Re(\gamma)>0\ ,\Re(\upsilon)>-1$$
$$\Re(\rho+\upsilon)>max\left[0\ \Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\right]$$

then there hold the formula:

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{\upsilon}]\right)\right)(x)=\frac{x^{\rho-\alpha-\alpha'+\gamma-1}\Gamma(\xi)}{\Gamma(\delta)}$$
$$\times\ {}_4\psi_4\left[\begin{matrix}(\rho+\gamma-\alpha-\alpha'-\beta,\upsilon),(\rho+\beta'-\alpha',\upsilon),(\delta,q),(1,1)\\(\rho+\gamma-\alpha-\alpha',\upsilon),(\rho+\gamma-\alpha'-\beta,\upsilon),(\rho+\beta',\upsilon),(\xi,p)\end{matrix}\middle|(cx)^{\upsilon}\right] \tag{2.1}$$

Proof: On using (1.4) and writing the function in the series form, the left hand side of (2.1), leads to

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{\upsilon}]\right)\right)(x)=\sum_{n=0}^{\infty}\frac{(\delta)_{qn}(c)^{\upsilon n}}{\Gamma(\nu n+\rho)(\xi)_{pn}}\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho+\upsilon n-1}\right)(x). \tag{2.2}$$

Now, upon using the image formula (1.10), which is valid under the conditions stated with Theorem 2.1, we get

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{\upsilon}]\right)\right)(x)=\frac{x^{\rho-\alpha-\alpha'+\gamma-1}\Gamma(\xi)}{\Gamma(\delta)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta+qn)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta+\upsilon n)}{\Gamma(\xi+pn)\Gamma(\rho+\gamma-\alpha-\alpha'+\upsilon n)}$$
$$\times\frac{\Gamma(\rho+\beta'-\alpha'+\upsilon n)\Gamma(1+n)}{\Gamma(\rho+\gamma-\alpha'-\beta+\upsilon n)\Gamma(\rho+\beta'+\upsilon n)}\frac{((cx)^{\upsilon})^{n}}{n!}. \tag{2.3}$$

Interpreting the right-hand side of the above equation, in view of the definition (1.5), we arrive at the result (2.1).

Theorem 2.2. Let $\alpha,\alpha',\beta,\beta',\gamma,\upsilon,\rho,\mu \in \mathrm{C}$ and $p,q,\nu > 0$, $q \le \Re(\nu)+p$, such that

$$\Re(\gamma)>0, \Re(\upsilon)>-1,$$
$$\Re(\rho-\upsilon)<1+min[\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)],$$

then the following formula holds true:

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\rho-\mu}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{-\upsilon}]\right)\right)(x)=\frac{x^{-\rho-\mu-\alpha-\alpha'+\gamma}\Gamma(\xi)}{\Gamma(\delta)}$$
$$\times\ {}_5\psi_5\left[\begin{matrix}(\alpha+\alpha'-\gamma+\mu+\rho,\upsilon),(\alpha+\beta'-\gamma+\mu+\rho,\upsilon),(\mu-\beta+\rho,\upsilon),(\delta,q),(1,1)\\(\mu+\rho,\upsilon),(\alpha+\alpha'+\beta'-\gamma+\mu+\rho,\upsilon),(\alpha-\beta+\mu+\rho,\upsilon),(\rho,\upsilon),(\xi,p)\end{matrix}\middle|(cx)^{-\upsilon}\right]. \tag{2.4}$$

Proof: By using (1.4), the left had side of (2.4), can be written as:

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\rho-\mu}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{-\upsilon}]\right)\right)(x)=\sum_{n=0}^{\infty}\frac{(\delta)_{qn}(c)^{-\upsilon n}}{\Gamma(\nu n+\rho)(\xi)_{pn}}\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{-\rho-\mu-\upsilon n}\right)(x), \tag{2.5}$$

which on using the image formula (1.11), arrive at

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}\left(t^{-\rho-\mu}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{-\upsilon}]\right)\right)(x)=\frac{x^{-\rho-\mu-\alpha-\alpha'+\gamma}\Gamma(\xi)}{\Gamma(\delta)}$$
$$\times\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+\alpha'-\gamma+\mu+\rho+\upsilon n)\Gamma(\alpha+\beta'-\gamma+\mu+\rho+\upsilon n)}{\Gamma(\mu+\rho+\upsilon n)\ \Gamma(\alpha+\alpha'+\beta'-\gamma+\mu+\rho+\upsilon n)}$$
$$\times\frac{\Gamma(\alpha-\beta+\mu+\upsilon n)\Gamma(\delta+qn)\ \Gamma(1+r)}{\Gamma(\alpha-\beta+\mu+\rho+\upsilon n)\ \Gamma(\rho+\upsilon n)\Gamma(\xi+pn)}\frac{((cx)^{-\upsilon})^n}{n!} \tag{2.6}$$

Interpreting the right-hand side of the above equation, in view of the definition (1.5), we arrive at the result (2.4).

3. SPECIAL CASES

In this section, we consider some special cases of the main results derived in the preceding section. If we set $\alpha'=0$ in the operators (1.8) and (1.9), then we have the following known identities:

$$\left(I_{0,+}^{\alpha+\beta,0,-\eta,\beta',\alpha}f\right)(x)=\left(I_{0,+}^{\alpha,\beta,\eta}f\right)(x). \tag{3.1}$$

$$\left(I_{0,-}^{\alpha+\beta,0,-\eta,\beta',\alpha}f\right)(x)=\left(I_{0,-}^{\alpha,\beta,\eta}f\right)(x). \tag{3.2}$$

where the hypergeometric operators, appeared in the right hand side are due to Saigo[10], defined as:

$$\left(I_{0,+}^{\alpha,\beta,\eta}f\right)(x)=\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\ {}_2F_1(\alpha+\beta,-\eta;\alpha;1-t/x)f(t)dt, \tag{3.3}$$

$$\left(I_{0,-}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}\, t^{-\alpha-\beta}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-x/t\right)f(t)dt. \tag{3.4}$$

Therefore, if we set $\alpha' = 0, \beta = -\eta, \gamma = \alpha$ and replace α by $\alpha+\beta$ in (2.1) and (2.2), we get the following results, involving the left and right hand sided Saigo type integral operators:

Corollary 3.1. Let $\alpha,\beta,\eta,\delta,\xi,\rho \in \mathbb{C}$ and $p,q,\nu > 0$, $q \le \Re(\nu)+p$, $\Re(\alpha)>0$, $\Re(\rho+\eta-\beta)>0$, then there hold the formula:

$$\left(I_{0,+}^{\alpha,\beta,\eta}\left(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{\upsilon}]\right)\right)(x) = \frac{x^{\rho-\beta-1}\Gamma(\xi)}{\Gamma(\delta)} \times {}_3\psi_3\left[\begin{matrix}(\rho+\eta-\beta,\upsilon),\ (\delta,\ q),\ (1,\ 1)\\ (\rho-\beta,\upsilon),\ (\rho+\alpha+\eta,\upsilon),\ (\xi,\ p)\end{matrix}\middle| (cx)^{\upsilon}\right] \tag{3.5}$$

Corollary 3.2. Let $\alpha,\beta,\eta,\delta,\xi,\rho \in \mathbb{C}$ and $p,q,\nu > 0$, $q \le \Re(\nu)+p$, $\Re(\alpha+\rho) > \max[-\Re(\beta),\Re(\eta)]$, then the following formula hold:

$$\left(I_{0,-}^{\alpha,\beta,\eta}\left(t^{-\mu-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{-\upsilon}]\right)\right)(x) = \frac{x^{-\rho-\mu-\beta}\Gamma(\xi)}{\Gamma(\delta)} \times {}_4\psi_4\left[\begin{matrix}(\rho+\mu+\beta,\upsilon),\ (\rho+\mu+\eta,\upsilon), (\delta,q),\ (1,1)\\ (\rho,\upsilon),\ (\rho+\mu,\upsilon),\ (\rho+\mu+\alpha+\beta+\eta,\upsilon), (\xi,p)\end{matrix}\middle| (cx)^{-\upsilon}\right]. \tag{3.6}$$

Further, if we follow results of Corollaries 3.1 and 3.2, when $\beta = -\alpha$, we arrive at the following results involving left and right sided Riemann-Liouville fractional integration operator.

Corollary 3.3. If $\alpha,\delta,\xi,\rho \in \mathbb{C}$ and $p,q,\nu>0$, $q \le \Re(\nu)+p$, $\Re(\alpha)>0$, $\Re(\rho)>0$, then we have

$$\left(I_{0,+}^{\alpha}\left(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{\upsilon}]\right)\right)(x) = \frac{x^{\rho+\alpha-1}\Gamma(\xi)}{\Gamma(\delta)}\,{}_2\psi_2\left[\begin{matrix}(\delta,\ q),\ (1,\ 1)\\ (\rho+\alpha,\upsilon),\ (\xi,\ p)\end{matrix}\middle| (cx)^{\upsilon}\right]. \tag{3.7}$$

Remark 1. If we set $\xi = p = q = 1$ in equation (3.7), we get the known result given by Saxena and Saigo [15].

Corollary 3.4. Let $\alpha,\delta,\xi,\rho \in \mathbb{C}$ and $p,q,\nu>0$, $q \le \Re(\nu)+p$, $\Re(\alpha+\rho) > \max[-\Re(-\alpha)]$, then

$$\left(W_{0,-}^{\alpha}\left(t^{-\mu-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{-\upsilon}]\right)\right)(x) = \frac{x^{-\rho-\mu-\alpha}\Gamma(\xi)}{\Gamma(\delta)}\,{}_3\psi_3\left[\begin{matrix}(\rho+\mu-\alpha,\upsilon), (\delta,q),\ (1,1)\\ (\rho,\upsilon), (\rho+\mu,\upsilon), (\xi,p)\end{matrix}\middle| (cx)^{-\upsilon}\right] \tag{3.8}$$

Remark 2. If we set $\xi = p = q = 1, \mu = \alpha$ in equation (3.8), we get the known result given by Saxena and Saigo [15].

Finally, if we follow Corollaries 3.1 and 3.2, in respective case $\beta = 0$, then we arrive at the following corollary concerning left and right sided Erdélyi-Kober fractional integration operators.

Corollary 3.5. Let $\alpha, \delta, \xi, \rho \in \mathbb{C}$ and $p, q, \nu > 0$, $q \le \Re(\nu) + p$, $\Re(\alpha) > 0$,$\Re(\rho + \eta) > 0$, then there hold the formula:

$$\left(E_{0,+}^{\alpha,\eta}\left(t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{\upsilon}]\right)\right)(x) = \frac{x^{\rho-1}\Gamma(\xi)}{\Gamma(\delta)}\,{}_3\psi_3\left[\begin{matrix}(\rho+\eta,\,\upsilon),\ (\delta,\ q),\ (1,\ 1)\\ (\rho,\,\upsilon),\ (\rho+\alpha+\eta,\upsilon),\ (\xi,\ p)\end{matrix}\middle|\,(cx)^{\upsilon}\right]. \tag{3.9}$$

Corollary 3.6. Let $\alpha, \delta, \xi, \rho \in \mathbb{C}$ and $p, q, \nu > 0$, $q \le \Re(\nu) + p$, $\Re(\alpha + \rho) > \max[\Re(\eta)]$, then there hold the formula

$$\left(K_{0,-}^{\alpha,\eta}\left(t^{-\mu-\rho}E_{\nu,\rho,p}^{\delta,\xi,q}[ct^{-\upsilon}]\right)\right)(x) = \frac{x^{-\rho-\mu}\Gamma(\xi)}{\Gamma(\delta)}\times{}_4\psi_4\left[\begin{matrix}(\rho+\mu,\,\upsilon),\,(\rho+\mu+\eta,\,\upsilon),\,(\delta,\,q),\ (1,1)\\ (\rho,\,\upsilon),\,(\rho+\mu,\,\upsilon),\,(\rho+\mu+\alpha+\eta,\,\upsilon),\,(\xi,p)\end{matrix}\middle|\,(cx)^{-\upsilon}\right] \tag{3.10}$$

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