ORIGINAL PAPER

ON BILATERAL GENERATING FUNCTIONS INVOLVING MODIFIED JACOBI POLYNOMIALS

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Abstract. *In this note, we have obtained some novel results on bilateral generating relations involving* $P_n^{(\alpha,\beta-\alpha)}(x)$*, modified Jacobi polynomials by group-theoretic method. In fact, in section 1, we have introduced a linear partial differential operator R which do not seem to have appeared in the earlier investigations and then we obtained the extended form of the group generated by R. Finally, in section 2, we have obtained a novel generating relation involving the polynomials under consideration with the help of which, we have proved a general theorem on bilateral generating relations of* $P_n^{(\alpha,\beta-\alpha)}(x)$.

Keywords: *Bilateral generating relation, Jacobi polynomials.*

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1. INTRODUCTION

Special functions are the solutions of a wide class of mathematically and physically relevant functional equations. Generating functions play a large role in the study of special functions. There are various methods of obtaining generating functions. But it has been found that group-theoretic method of obtaining generating functions is much potent one in comparison to analytic method. The study of special functions, in particular, generating functions of special functions by group-theoretic method was originally introduced by L.Weisner [1] while studying generating functions of Hypergeometric function in the year 1955. From seventies and onwards (i.e. just after the publication of the monograph "obtaining generating functions" by E.B.McBride [2]) of the last century, Weisner's group theoretic method has been utilized by researchers while deriving generating functions of various special functions.

In the present article we have obtained some novel bilateral generating relations of $P_n^{(\alpha,\beta-\alpha)}(x)$, a modification of Jacobi polynomials, by group-theoretic method, where $P_n^{(\alpha,\beta)}(x)$ is defined by [3]:

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(1.1) $$P_n^{(\alpha,\,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left[\begin{matrix} -n,\ 1+\alpha+\beta+n\ ; \\ \\ 1+\alpha\ ; \end{matrix}\ \frac{1-x}{2}\right].$$

The main result of our investigation is stated in the form of the following theorem. For previous works on bilateral generating functions of Jacobi / modified Jacobi polynomials, one may refer to the works [5-9].

Theorem 1. If there exists a unilateral generating relation of the form:

(1.2) $$G(x,w) = \sum_{\alpha=0}^{\infty} a_n P_n^{(\alpha,\beta-\alpha)}(x)\ \ w^{\alpha}$$

then

(1.3) $$(1-w)^{\beta}\ G\left(x-(1+x)w,\ \frac{wt}{1-w}\right) = \sum_{\alpha=0}^{\infty} w^{\alpha}\ P_n^{(\alpha,\beta-\alpha)}(x)\sigma_{\alpha}(t)\,,$$

where

(1.4) $$\sigma_{\alpha}(t) = \sum_{p=0}^{\alpha} \frac{1}{(\alpha-p)!} a_p(-\beta-n+p)_{\alpha-p}\ t^{p}.$$

The importance of the above theorems lies in the fact that whenever one knows a unilateral generating relation of type (1.2), the corresponding bilateral generating relation can at once be written down from (1.3). Thus a large number of bilateral generating relations can be obtained by attributing different values to a_n in (1.2).

2. DERIVATION OF THE OPERATOR AND ITS EXTENDED FORM OF THE GROUP

At first we seek the following first order linear partial differential operator:

(2.1) $$R = R_1\frac{\partial}{\partial x} + R_2\frac{\partial}{\partial y} + R_0$$

such that

(2.2) $$R(P_n^{(\alpha,\,\beta-\alpha)}(x)y^{\alpha}) = a_{\alpha}P_n^{(\alpha+1,\beta-\alpha-1)}(x)y^{\alpha+1}\ ,$$

where $R_i\,(i=0,1,2)$ are functions of $x,\ y$ but independent of α and a_{α} is a function of $n,\ \alpha,\ \beta$.

Noticing the following differential recurrence relation [3]:

(2.3) $$\frac{d}{dx}\left(P_n^{(\alpha,\,\beta-\alpha)}(x)\right)$$
$$= \frac{1}{1-x^2}\left[(\beta-\alpha+n)(1-x)P_n^{(\alpha+1,\,\beta-\alpha-1)}(x)-(1-x)(\beta-\alpha)P_n^{(\alpha,\,\beta-\alpha)}(x)\right],$$

we define

(2.4) $$R = y(1+x)\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} + \beta y$$

such that

(2.5) $$R(P_n^{(\alpha,\,\beta-\alpha)}(x)y^{\alpha}) = (\beta-\alpha+n)P_n^{(\alpha+1,\beta-\alpha-1)}(x)y^{\alpha+1}.$$

We now proceed to find the extended form of the group generated by R i.e. we shall find $e^{wR}f(x,y)$ where $f(x,y)$ is arbitrary function and w is arbitrary constant, real or complex.

If $\varphi(x,y)$ be a solution of $R\varphi(x,y)=0$ and if we transform the operator R to E such that

$$E = R_1\frac{\partial}{\partial x} + R_2\frac{\partial}{\partial y},$$

then

$$E = \phi^{-1}(x,y)\,R\,\phi(x,y)$$

i.e.
$$R = \phi(x,y)E\phi^{-1}(x,y).$$

Therefore, we have

$$e^{wR}f(x,y) = e^{w\phi(x,y)E\phi^{-1}(x,y)}f(x,y)$$
$$= \phi(x,y)e^{wE}\left(\phi^{-1}(x,y)\,f(x,y)\right).$$

Finally, we choose new variables X, Y so that the operator E is transformed into the operator $D \equiv \dfrac{\partial}{\partial X}$. Under this change of variables, let $\varphi^{-1}(x,y)f(x,y)$ be transformed into $F(X,Y)$.

Therefore, by Taylor's theorem, we get

$$e^{wR}f(x,y) = \varphi(x,y)\,e^{wD}\left(F(X,Y)\right)$$
$$= \varphi(x,y)F(X+w,Y)$$
$$= \varphi(x,y)\,g(x,y),$$

supposing that $F(X+w,Y)$ is transformed into $g(x,y)$ by inverse substitution.

By the method out-lined, we shall compute $e^{wR}f(x,y)$ where

$$R = y(1+x)\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} + \beta y.$$

Let $\varphi(x,y)$ be a function such that $R\varphi(x,y)=0$. Then on solving we get

$$\phi(x,y) = y^{-1}(1+x)^{-1-\beta}.$$

Therefore $E = \varphi^{-1}(x,y)R\varphi\,(x,y) = y(1+x)\dfrac{\partial}{\partial x} - y^2\dfrac{\partial}{\partial y}$.

Now let X, Y be a set of new variables for which

(2.6) $$EX = 1, \quad EY = 0,$$

so that E reduces to $\dfrac{\partial}{\partial X}$.

Now solving (2.6), we get a set of solutions as follows:

$$X = \frac{1}{y},\;\; Y = y(1+x)$$

From which we get

$$x = XY - 1,\;\; y = \frac{1}{X}$$

Recalling $R = \varphi\,E\,\varphi^{-1}$ where $\varphi(x,y) = y^{-1}(1+x)^{-1-\beta}$, we get

$$e^{wR} f(x,y) = e^{w\,\phi E\,\phi^{-1}} f(x,y)$$
$$= y^{-1}(1+x)^{-1-\beta}\, e^{wR}\left[y(1+x)^{1+\beta}\; f(x,y)\right].$$

Now the transformations $x = XY - 1,\ y = \dfrac{1}{X}$ will transform E into $D \equiv \left(\dfrac{\partial}{\partial X}\right)$.

Making the substitutions and applying the Taylor's theorem, we get

$$e^{wE}\left(y(1+x)^{1+\beta}\; f(x,y)\right) = e^{wD} X^{\beta}\, Y^{1+\beta} f\left(XY - 1, \frac{1}{X}\right)$$

$$= e^{wD}(X+w)^{\beta}\, Y^{1+\beta} f\left((X+w)Y - 1, \frac{1}{X+w}\right).$$

Finally substituting $X = \dfrac{1}{y},\ Y = y(1+x)$, we get

$$e^{wE}\left(y(1+x)^{1+\beta}\; f(x,y)\right) = \left(\frac{1}{y} + w\right)^{\beta} \left[y(1+x)\right]^{1+\beta} f\left(\left(\frac{1}{y} + w\right) y(1+x) - 1, \frac{1}{\frac{1}{y} + w}\right)$$

$$= (1+wy)^{\beta}\; y(1+x)^{1+\beta} f\left(x + (1+x)yw, \frac{y}{1+yw}\right).$$

Therefore, we get

$$(2.7)\qquad e^{wR} f(x,y) = y^{-1}(1+x)^{-1-\beta}\, e^{wR}\left[y(1+x)^{1+\beta}\; f(x,y)\right]$$

$$= y^{-1}(1+x)^{-1-\beta}(1+wy)^{\beta}\; y(1+x)^{1+\beta} f\left(x + (1+x)yw, \frac{y}{1+yw}\right)$$

$$= (1+wy)^{\beta} f\left(x + (1+x)yw, \frac{y}{1+yw}\right).$$

APPLICATION OF THE OPERATOR

Now, writing $f(x,y) = P_n^{(\alpha,\beta-\alpha)}(x) y^{\alpha}$ in (2.7), we get

$$(2.8)\qquad e^{wR}\ \left(P_n^{(\alpha,\beta-\alpha)}(x) y^{\alpha}\right) = (1+wy)^{\beta-\alpha}\, P_n^{(\alpha,\beta-\alpha)}\left(x + (1+x)wy\right) y^{\alpha}.$$

Again, on the other hand with the help of (2.5), we have

$$(2.9)\qquad e^{wR}\left(P_n^{(\alpha,\beta-\alpha)}(x)\, y^{\alpha}\right)$$

$$= \sum_{p=0}^{\infty} \frac{(w)^p}{p!} R^p\left(P_n^{(\alpha,\beta-\alpha)}(x)\, y^{\alpha}\right)$$

$$= \sum_{p=0}^{\infty} \frac{(-w)^p}{p!} (-\beta - n + \alpha)_p\, P_n^{(\alpha+p,\beta-\alpha-p)}(x)\, y^{\alpha+p}.$$

Equating (2.8) & (2.9) and putting, $y = -1$, we get

$$(2.10)\qquad (1-w)^{\beta-\alpha}\, P_n^{(\alpha,\beta-\alpha)}\left(x - (1+x)w\right) = \sum_{p=0}^{\infty} \frac{w^p}{p!} (-\beta - n + \alpha)_p\, P_n^{(\alpha+p,\beta-\alpha-p)}(x)$$

We now proceed to prove the Theorem 1 by using the above generating relation.

Proof of the Theorem 1: Now the right hand side of (1.3),

$$\sum_{\alpha=0}^{\infty} w^{\alpha} P_n^{(\alpha,\beta-\alpha)}(x)\sigma_{\alpha}(t)$$

$$= \sum_{\alpha=0}^{\infty} w^{\alpha} P_n^{(\alpha,\beta-\alpha)}(x) \sum_{p=0}^{\alpha}\frac{1}{(\alpha-p)!} a_p(-\beta-n+p)_{\alpha-p}\, t^p \quad \text{[from (1.4)]}$$

$$= \sum_{\alpha=0}^{\infty}\sum_{p=0}^{\infty} w^{\alpha+p} P_n^{(\alpha+p,\beta-\alpha-p)}(x)\frac{1}{\alpha!} a_p(-\beta-n+p)_{\alpha}\, t^p$$

$$= \sum_{p=0}^{\infty} w^p a_p\, t^p \sum_{\alpha=0}^{\infty}\frac{w^{\alpha}}{\alpha!}(-\beta-n+p)_{\alpha}\quad P_n^{(\alpha+p,\beta-\alpha-p)}(x)$$

$$= \sum_{p=0}^{\infty} a_p\,(wt)^p\,(1-w)^{\beta-p} P_n^{(p,\beta-p)}(x-(1+x)w) \quad \text{[from (2.10)]}$$

$$= \sum_{p=0}^{\infty} a_p\, P_n^{(p,\beta-p)}\big(x-(1+x)w\big)\left(\frac{wt}{1+w}\right)^p (1-w)^{\beta}$$

$$= (1-w)^{\beta}\, G\!\left(x-(1+x)w, \frac{wt}{1-w}\right). \text{ [from (1.2)]}$$

=Left hand side of (1.3),

which is Theorem 1.

Finally, we would like to point it out that the Theorem 1 can be proved as follows by the direct application of the operator R using the method as discussed in [4].

We first consider the following unilateral generating relation of the form

$$G(x,w) = \sum_{\alpha=0}^{\infty} a_{\alpha} P_n^{(\alpha,\beta-\alpha)}(x)\, w^{\alpha}. \tag{2.11}$$

Replacing *w* by *wyt* in (2.2.16) and then operating e^{wR} on both sides, we get

$$e^{wR}\, G(x,wyt) = e^{wR}\left(\sum_{\alpha=0}^{\infty} a_{\alpha} P_n^{(\alpha,\beta-\alpha)}(x)\,(wyt)^{\alpha}\right). \tag{2.12}$$

The left member of (2.12), with the help of (2.7), becomes

$$(1+wy)^{\beta}\, G\!\left(x+(1+x)wy, \frac{wyt}{1+wy}\right). \tag{2.13}$$

The right member of (2.12), with the help of (2.5), becomes

$$\sum_{\alpha=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-w)^p}{p!} a_{\alpha}\,(-\beta-n+\alpha)_p\, P_n^{(\alpha+p,\,\beta-\alpha-p)}(x)\,(wt)^{\alpha}\, y^{\alpha+p}. \tag{2.14}$$

Equating (2.13) and (2.14), and then putting $y = 1$, we get

$$(1+w)^{\beta}\, G\!\left(x+(1+x)w, \frac{wt}{1+w}\right) \tag{2.15}$$

$$= \sum_{\alpha=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-w)^p}{p!} a_{\alpha}\,(-\beta-n+\alpha)_p\, P_n^{(\alpha+p,\,\beta-\alpha-p)}(x)\,(wt)^{\alpha}.$$

Now replacing *w* by –*w* and *t* by –*t* and simplifying we get

$$(1-w)^{\beta}\, G\!\left(x-(1+x)wy, \frac{wt}{1-w}\right) = \sum_{n=0}^{\infty} w^{\alpha} P_n^{(\alpha,\beta-\alpha)}(x)\sigma_{\alpha}(t),$$

where

$$\sigma_\alpha(t) = \sum_{p=0}^{\alpha} \frac{1}{(\alpha - p)!} a_p (-\beta - n + p)_{\alpha - p}\ t^p\ .$$

This completes the proof of the Theorem 1.

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REFERENCES

[1] Weisner, L., Group theoretic origins of certain generating functions, *Pacific J. Math.*, **5**, 1033, 1955.
[2] McBride, E.D., *Obtaining generating functions,* Springer-Verlag, New York, Heidelberg, Berlin, 1971.
[3] Rainville, E.D., *Special functions,* Macmillan, New York, 1960.
[4] Chongdar, A.K., Chatterjea, S.K., *Bull.Cal. Math. Soc*., **73**, 127, 1981.
[5] Chongdar, A.K., Comn. Fac. Sci.Univ. Ankara Soc. A, *Math. Statist*, **41**(1-2), 169, 1994.
[6] Chongdar, A.K., *Bull. Cal. Math. Soc*., **78,** 363, 1986.
[7] Ghosh, B., Some theorems on more general generating functions of Jacobi polynomials, *Gadnita*, **39**(1-2), 55, 1988.
[8] Das, S., On partial differential operators for Jacobi Polynomials, *Pure Math. Manuscripts* **3**, 95, 1984.
[9] Bera, C.S., Chongdar, A.K., *Ultra Scientist*, **24**(1), 1, 2012.