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WEIGHTED CORRELATION ON EXTENDED FRACTIONAL FOURIER TRANSFORM DOMAIN

ALKA S. GUDADHE[1], NAVEEN REDDY A.[2]

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Abstract. *In this paper, we defined the correlations based on the extended fractional Fourier transform domain. The correlation uses two functions to produce a third function in its own form. This third function is called the correlation of the two functions. Properties of correlations on the extended fractional Fourier transform domain are also presented.*

Keywords: *Extended fractional Fourier transform, fractional Fourier transform, cross-correlation and auto-correlation.*

1. INTRODUCTION

The fractional Fourier transform (FrFT) is a generalization of the classical Fourier transform (FT) into fractional domains, is introduced way back in 1920's, but remained largely unknown until the work of Namias [5] in 1980. It is used in many areas of communication systems, signal processing and optics [7, 8]. Number of applications of his FrFT can be found in [1, 8, 9].

Extended fractional Fourier transform (EFrFT) is the generalization of FrFT can be seen in Jianwen Hua et. al. [4] with two more parameters as,

$$F_{a,b}^{\alpha}\left[f(t)\right](u)=F_{a,b}^{\alpha}(u)=\int_{-\infty}^{\infty}f(t)K_{a,b}^{\alpha}(t,u)d\ \ t \tag{1.1}$$

where $K_{a,b}^{\alpha}(t,u)=e^{i\pi\left[\left(a^2t^2+b^2u^2\right)\cot\alpha-2abtu\csc\alpha\right]}$.

In [3] we are introduced the properties on EFrFT domain, and in [2] we have enhanced the concept of convolution to the domain of EFrFT.

In this paper, we have defined the correlations of EFrFT and proved some properties on it.

[1] Govt. Vidarbha Institute of Science and Humanities, Amravati. (M. S.), India.
E-mail: alka.gudadhe@gmail.com.
[2] Vidyabharti Mahavidyalaya, Amravati. (M. S.), India. E-mail: anreddyssv@gmail.com.

2. CROSS-CORRELATION THEOREM FOR EFrFT:

Definition 2.1: For two functions *f(t)* and *g(t)* of extended fractional Fourier transform, the weighted correlation operation can be defined as,

$$h(t)=(f\oplus g)(t)=\int_{-\infty}^{\infty} f(\tau)g(t+\tau)e^{2i\pi a^2\tau(t+\tau)\cot\alpha}d\tau \tag{2.1}$$

where $\oplus$ is the cross correlation operation for the extended fractional Fourier transform.

Theorem 2.1. Let $h(t)=(f\oplus g)(t)$ be the weighted correlation and $F_{a,b}^{\alpha}$, $G_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transforms of functions *f*, *g* and *h* respectively, then

$$H_{a,b}^{\alpha}(u)=e^{-i\pi b^2u^2\cot\alpha}F_{a,b}^{\alpha}(-u)G_{a,b}^{\alpha}(u) \tag{2.2}$$

Proof: By (1.1) and (2.1)

$$H_{a,b}^{\alpha}(u)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)g(t+\tau)e^{2i\pi a^2\tau(t+\tau)\cot\alpha}d\tau.e^{i\pi\left[\left(a^2t^2+b^2u^2\right)\cot\alpha-2abtu\csc\alpha\right]}dt$$

Replacing $t+\tau=v\Rightarrow t=v-\tau$ and $dt=dv$

$$H_{a,b}^{\alpha}(u)=e^{-i\pi b^2u^2\cot\alpha}F_{a,b}^{\alpha}(-u)G_{a,b}^{\alpha}(u)$$

Hence proved.

3. PROPERTIES OF CROSS CORRELATION OF EFrFT

3.1. Commutative Law: If $F_{a,b}^{\alpha}$, $G_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transform of functions *f*, *g* and *h* respectively and

$$h(t)=(f\oplus g)(t)=\int_{-\infty}^{\infty} f(\tau)g(t+\tau)e^{2i\pi a^2\tau(t+\tau)\cot\alpha}d\tau$$

weighted cross correlation of EFrFT of *f* and *g* then

$$(f\oplus g)(t)\neq(g\oplus f)(t)$$

Proof: Let $z(t)=(g\oplus f)(t)$

By (1.1),

$$Z_{a,b}^{\alpha}(u)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau)f(t+\tau)e^{2i\pi a^2\tau(t+\tau)\cot\alpha}e^{i\pi\left[\left(a^2t^2+b^2u^2\right)\cot\alpha-2abtu\csc\alpha\right]}d\tau dt$$

Replace $t+\tau=v$ then $t=v-\tau$ and $dt=dv$.

Therefore

$$Z_{a,b}^{\alpha}(u)=e^{-i\pi b^2u^2\cot\alpha}G_{a,b}^{\alpha}(-u)F_{a,b}^{\alpha}(u) \tag{3.1}$$

Therefore from (2.2) and (3.1)

$$H_{a,b}^{\alpha}(u) \neq Z_{a,b}^{\alpha}(u)$$

That is $(f \oplus g)(t) \neq (g \oplus f)(t)$

3.2. Associative Law: If $F_{a,b}^{\alpha}$, $G_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transform of functions *f*, *g* and *h* respectively and

$$h(t) = (f \oplus g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t+\tau) e^{2i\pi a^2 \tau(t+\tau)\cot\alpha} d\tau$$

then

$$[(f \oplus g) \oplus y](t) \neq [f \oplus (g \oplus y)](t)$$

Proof: The proof is simple and hence omitted.

3.3. Distributive Law: If $F_{a,b}^{\alpha}$, $G_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transform of functions *f*, *g* and *h* respectively and

$$h(t) = (f \oplus g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t+\tau) e^{2i\pi a^2 \tau(t+\tau)\cot\alpha} d\tau$$

then

$$[f \oplus (g+y)](t) = [(f \oplus g) + (f \oplus y)](t)$$

Proof: Let $z_1(t) = [f \oplus (g+y)](t)$

By (1.1), EFrFT of $z_1(t)$ will be,

$$Z_{1,a,b}^{\alpha}(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)[g(t+\tau) + y(t+\tau)] e^{2i\pi a^2 \tau(t+\tau)\cot\alpha} e^{i\pi(a^2t^2+b^2u^2)\cot\alpha - 2i\pi abtu\csc\alpha} d\tau dt$$

Substituting $t+\tau = w$ then $t = w-\tau$ and $dt = dw$, we obtain

$$Z_{1,a,b}^{\alpha}(u) = e^{-i\pi b^2u^2\cot\alpha} F_{a,b}^{\alpha}(-u)\left[G_{a,b}^{\alpha}(u) + Y_{a,b}^{\alpha}(u)\right] \tag{3.2}$$

Let $z_2(t) = [(f \oplus g) + (f \oplus y)](t) = (f \oplus g)(t) + (f \oplus y)(t)$

By (1.1), EFrFT of $z_2(t)$ will be,

$$Z_{2,a,b}^{\alpha}(u) = \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(\tau) g(t+\tau) e^{2i\pi a^2 \tau(t+\tau)\cot\alpha} d\tau + \int_{-\infty}^{\infty} f(\tau) y(t+\tau) e^{2i\pi a^2 \tau(t+\tau)\cot\alpha} d\tau\right\}$$
$$e^{i\pi(a^2t^2+b^2u^2)\cot\alpha - 2i\pi abtu\csc\alpha} dt$$

Substituting $t+\tau = w$ then $t = w-\tau$ and $dt = dw$, we obtain

$$Z_{2,a,b}^{\alpha}(u) = e^{-i\pi b^2u^2\cot\alpha} F_{a,b}^{\alpha}(-u)\left[G_{a,b}^{\alpha}(u) + Y_{a,b}^{\alpha}(u)\right] \tag{3.3}$$

Therefore from (3.2) and (3.3)

$$[f \oplus (g+y)](t) = [(f \oplus g) + (f \oplus y)](t)$$

Hence proved.

3.4. Evenness: If $F_{a,b}^{\alpha}$, $G_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transform of *f*, *g* and *h* respectively and

$$h(t) = (f \oplus g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t+\tau) e^{2i\pi a^2 \tau (t+\tau)\cot\alpha} d\tau$$

then

$$(f \oplus g)(t) = (g \oplus f)(-t) \neq (f \oplus g)(-t)$$

Proof: Let $h_1(t) = (f \oplus g)(t)$

By (2.2)

$$H_{1,a,b}^{\alpha}(u) = e^{-i\pi b^2 u^2 \cot\alpha} F_{a,b}^{\alpha}(-u) G_{a,b}^{\alpha}(u) \tag{3.4}$$

Let $h_2(t) = (f \oplus g)(-t) = \int_{-\infty}^{\infty} f(\tau) g(-t+\tau) e^{2i\pi a^2 \tau(-t+\tau)\cot\alpha} d\tau$

By (1.1)

$$H_{2,a,b}^{\alpha}(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau) g(-t+\tau) e^{2i\pi a^2 \tau(-t+\tau)\cot\alpha} d\tau . e^{i\pi\left[\left(a^2t^2+b^2u^2\right)\cot\alpha - 2abtu\csc\alpha\right]} dt$$

Replacing $\tau - t = -v \Rightarrow t = \tau + v$ and $dt = dv$

$$H_{2,a,b}^{\alpha}(u) = e^{-i\pi b^2 u^2 \cot\alpha} F_{a,b}^{\alpha}(u) G_{a,b}^{\alpha}(-u) \tag{3.5}$$

Let $h_3(t) = (g \oplus f)(-t)$

Therefore by (1.1)

$$H_{3,a,b}^{\alpha}(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau) f(\tau - t) e^{2i\pi a^2 \tau(\tau-t)\cot\alpha} e^{i\pi\left[\left(a^2t^2+b^2u^2\right)\cot\alpha - 2abtu\csc\alpha\right]} d\tau dt$$

Replace $\tau - t = -v$ then $t = v + \tau$ and $dt = dv$

$$H_{3,a,b}^{\alpha}(u) = e^{-i\pi b^2 u^2 \cot\alpha} G_{a,b}^{\alpha}(u) F_{a,b}^{\alpha}(-u) \tag{3.6}$$

By (3.4), (3.5) and (3.6)

$$(f \oplus g)(t) = (g \oplus f)(-t) \neq (f \oplus g)(-t)$$

4. AUTO-CORRELATION THEOREM FOR EFrFT:

Definition 4.1. For any functions $f(t)$ and $h(t)$, the weighted correlation operation can be defined as

$$h(t) = (f \square f)(t) = \int_{-\infty}^{\infty} f(\tau) f(t+\tau) e^{2i\pi a^2 \tau(t+\tau)\cot\alpha} d\tau \tag{4.1}$$

where $\square$ is the auto-correlation operation for the extended fractional Fourier transform.

Theorem 4.1. Let $h(t) = (f \square f)(t)$ is the weighted correlation and $F_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transforms of functions *f* and *h* respectively, then

$$H_{a,b}^{\alpha}(u) = e^{-i\pi b^2 u^2 \cot\alpha} F_{a,b}^{\alpha}(-u) F_{a,b}^{\alpha}(u) \tag{4.2}$$

Proof: By (1.1) and (4.1)

$$H_{a,b}^{\alpha}(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\tau)\, f(t+\tau)\, e^{2i\pi a^2 \tau (t+\tau)\cot\alpha} d\tau . e^{i\pi\left[\left(a^2t^2+b^2u^2\right)\cot\alpha - 2abtu\csc\alpha\right]} dt$$

Replacing $t+\tau = v \Rightarrow t = v-\tau$ and $dt = dv$

$$H_{a,b}^{\alpha}(u) = e^{-i\pi b^2u^2\cot\alpha} F_{a,b}^{\alpha}(-u) F_{a,b}^{\alpha}(u)$$

Hence proved.

5. PROPERTIES OF AUTO-CORRELATION OF EFrFT:

5.1. Commutative Law: If $F_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transform of functions *f* and *h* respectively where *h* is auto correlation

$$h(t) = (f \square f)(t) = \int_{-\infty}^{\infty} f(\tau)\, f(t+\tau)\, e^{2i\pi a^2 \tau (t+\tau)\cot\alpha} d\tau$$

then

$$(f \square f)(t) = (f \square f)(t)$$

Proof: It is obvious.

5.2. Associative Law: If $F_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transform of functions *f* and *h* respectively and

$$h(t) = (f \square f)(t) = \int_{-\infty}^{\infty} f(\tau)\, f(t+\tau)\, e^{2i\pi a^2 \tau (t+\tau)\cot\alpha} d\tau$$

then

$$\left[(f \square f) \square f\right](t) \neq \left[f \square (f \square f)\right](t)$$

Proof: By associative law of cross-correlation of EFrFT.

$$\left[(f \square f) \square f\right](t) \neq \left[f \square (f \square f)\right](t)$$

5.3. Distributive Law: If $F_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transform of functions *f* and *h* respectively and

$$h(t) = (f \square f)(t) = \int_{-\infty}^{\infty} f(\tau)\, f(t+\tau)\, e^{2i\pi a^2 \tau (t+\tau)\cot\alpha} d\tau$$

then

$$\left[f \square (f+f)\right](t) = \left[(f \square f) + (f \square f)\right](t)$$

Proof: By distributive law of cross-correlation of EFrFT.

$$\left[f \square (f+f)\right](t) = \left[(f \square f) + (f \square f)\right](t)$$

5.4. Evenness: If $F_{a,b}^{\alpha}$ and $H_{a,b}^{\alpha}$ denote the extended fractional Fourier transform of functions *f* and *h* respectively and

$$h(t) = (f \square f)(t) = \int_{-\infty}^{\infty} f(\tau)\, f(t+\tau)\, e^{2i\pi a^2 \tau (t+\tau)\cot\alpha} d\tau$$

then

$$(f \square f)(t) = (f \square f)(-t)$$

Proof: By evenness of cross-correlation of EFrFT.

$$\left(f \star f\right)(t) = \left(f \star f\right)(-t)$$

6. CONCLUSIONS

In this paper, we have defined the weighted cross-correlation and auto correlation in the extended fractional Fourier transform domain. We have also established the cross correlation theorem and proved some related properties. It should also be noted that the theorem and properties are converted to the case of FrFt if we put $a=b=1$ and moreover they are transformed to the case of conventional FT, if further α is taken as $\pi/2$. All these properties and theorem can be utilized in the theory of signal processing.

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