ORIGINAL PAPER

ON INTEGRAL FORMS OF SEVERAL INEQUALITIES

LOREDANA CIURDARIU[1]

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Abstract. *In this paper we give some integral forms of some refinements and counterparts of Radon's inequality using recent generalizations.*

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1. INTRODUCTION

We will recall the inequality of J. Radon which was published in [6].

For every real numbers $p > 0, x_k \geq 0, a_k > 0$ for $1 \leq k \leq n,$ we have the following inequality:

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \frac{(\sum_{k=1}^{n} x_k)^{p+1}}{(\sum_{k=1}^{n} a_k)^p}, p > 0.$$

In [7], the authors consider two n-tuples $a = (a_1, a_2, ..., a_n), b = (b_1, b_2, ..., b_n)$ where $ab = (a_1b_1, a_2b_2, ..., a_nb_n)$ and $a^m = (a_1^m, a_2^m, ..., a_n^m),$ for any real number *m*.

Then $a > 0$ and $b > 0$ if $a_i > 0$ and $b_i > 0$ for every $1 < i < n.$

We consider the expression:

$$\Delta_n^{[p]}(a;b) = \sum_{i=1}^{n} \frac{a_i^p}{b_i^{p-1}} - \frac{(\sum_{i=1}^{n} a_i)^p}{(\sum_{i=1}^{n} b_i)^{p-1}},$$

for real number $p > 1$ and for n-tuples $a \geq 0$ and $b \geq 0.$

Then the well-known Radon's inequality can be written as:

$$\Delta_n^{[p]}(a;b) \geq 0.$$

Theorem 1. ([7]) For every $n \geq 2, p \geq 1, a_k \geq 0, b_k > 0, 1 \leq k \leq n,$ the following inequality hold:

[1] Politehnica University of Timisoara, Mathematics Department, 300006, Timisoara, Romania.
E-mail: siloredc@yahoo.com.

$$0 \le \Delta_n^{[p]}(a;b) \le p\left(\Delta_n^{[p]}(a;b) - \frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}\Delta_n^{[p-1]}(a;b)\right)$$

and

$$0 \le \Delta_n^{[p]}(a;b) \le p(M-m)(M^{p-1}-m^{p-1})\left(\sum_{i=1}^n b_i\right)$$

where $m \le \frac{a_i}{b_i} \le M$, for $i = 1,...,n.$

It is necessary to recall also Theorem 2.9 and Theorem 2.7 from [7].

Theorem 2. ([7]) There is the inequality

$$0 \le \Delta_n^{[p]}(a;b) \le \frac{\left[(M+m)\sum_{i=1}^n b_i - \sum_{i=1}^n a_i\right]^p}{\left(\sum_{i=1}^n b_i\right)^{p-1}} - \frac{(M+m)^p}{2^{p-1}}\left(\sum_{i=1}^n b_i\right) + \left(\sum_{i=1}^n \frac{a_i^p}{b_i^{p-1}}\right),$$

where $m \le \frac{a_i}{b_i} \le M$, $a_i \ge 0, b_i > 0, 1 \le i \le n, p \ge 1, n \ge 2.$

Theorem 3. ([7]) For $n \ge 2, p \ge 1,$ we have the following inequalities:

$$\Delta_n^{[p]}(a;b) \ge \max_{1 \le i < j \le n}\left[\frac{a_i^p}{b_i^{p-1}} + \frac{a_j^p}{b_j^{p-1}} - \frac{(a_i+a_j)^p}{(b_i+b_j)^{p-1}}\right],$$

and

$$0 \le \Delta_n^{[p]}(a;b) \le \left[M^p + m^p - \frac{(M+m)^p}{2^{p-1}}\right]\left(\sum_{i=1}^n b_i\right),$$

where $m \le \frac{a_i}{b_i} \le M$, $a_i \ge 0, b_i > 0, 1 \le i \le n.$

We need the following result from [7], which will be used also below, in the next section.

Theorem 4. If $a = (a_1, a_2, ..., a_n)$ and $b = (b_1, b_2, ..., b_n)$ are n-tuples then we have the inequality:

$$\frac{p(p-1)m^{p-2}}{2\sum_{i=1}^n b_i}\sum_{1 \le i < j \le n}\frac{(a_i b_j - a_j b_i)^2}{b_i b_j} \le \Delta_n^{[p]}(a;b) \le$$

$$\le \frac{p(p-1)M^{p-2}}{2\sum_{i=1}^n b_i}\sum_{1 \le i < j \le n}\frac{(a_i b_j - a_j b_i)^2}{b_i b_j},$$

where $m \le \frac{a_i}{b_i} \le M$, $p > 1$, $a_i \ge 0, b_i > 0$ for $i = 1,...,n.$

2. INTEGRAL FORMS OF SEVERAL INEQUALITIES

Using the same techniques as in [1] we find the following integral form of the inequality (2.5) and (2.6) from Theorem 2.3, see [7].

Theorem 5. For every $n \ge 2, p \ge 1, f(x) \ge 0, g(x) > 0$ and if $f, g : [a,b] \to R_+$ are two continuous functions on $[a,b]$ with $m = \inf\limits_{x\in[a,b]} \frac{f(x)}{g(x)}$, $M = \sup\limits_{x\in[a,b]} \frac{f(x)}{g(x)}$ then we have:

$$0 \le \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} \le \frac{p}{4}(M-m)(M^{p-1} - m^{p-1})\int_a^b g(x)dx.$$

If $f, g : [a,b] \to R_+$ are two integrable functions on $[a,b]$ then

$$0 \le \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} \le$$

$$\le p\left[\int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} - \frac{\int_a^b f(x)dx}{\int_a^b g(x)dx}\left(\int_a^b \frac{(f(x))^{p-1}}{(g(x))^{p-2}} dx - \frac{\left(\int_a^b f(x)dx\right)^{p-1}}{\left(\int_a^b g(x)dx\right)^{p-2}}\right)\right].$$

Proof: Let $n \in N$ and $x_k = k + \frac{b-a}{n}, k \in \{0,1,...,n\}$. Using Theorem 2.3, see [7] we have

$$0 \le \sum_{k=1}^n \frac{(f(x_k))^p}{(g(x_k))^{p-1}} - \frac{\left(\sum_{k=1}^n f(x_k)\right)^p}{\left(\sum_{k=1}^n g(x_k)\right)^{p-1}} \le$$

$$\le p\left[\sum_{k=1}^n \frac{(f(x_k))^p}{(g(x_k))^{p-1}} - \frac{\left(\sum_{k=1}^n f(x_k)\right)^p}{\left(\sum_{k=1}^n g(x_k)\right)^{p-1}} - \frac{\sum_{k=1}^n f(x_k)}{\sum_{k=1}^n g(x_k)}\left(\sum_{k=1}^n \frac{(f(x_k))^{p-1}}{(g(x_k))^{p-2}} - \frac{\left(\sum_{k=1}^n f(x_k)\right)^{p-1}}{\left(\sum_{k=1}^n g(x_k)\right)^{p-2}}\right)\right]$$

and

$$0 \le \sum_{k=1}^{n} \frac{\left(f(x_k)\right)^p}{\left(g(x_k)\right)^{p-1}} - \frac{\left(\sum_{k=1}^{n} f(x_k)\right)^p}{\left(\sum_{k=1}^{n} g(x_k)\right)^{p-1}} \le \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})\left(\sum_{k=1}^{n} g(x_k)\right),$$

where $m \le \frac{a_i}{b_i} \le M$, for $i = 1,...,n.$

It results that

$$0 \le \sigma\left(\frac{f^p}{g^{p-1}}, \Delta_n, x_k\right) - \frac{\left(\sigma(f,\Delta_n,x_k)\right)^p}{\left(\sigma(g,\Delta_n,x_k)\right)^{p-1}} \le$$

$$\le p\left(\sigma\left(\frac{f^p}{g^{p-1}}, \Delta_n, x_k\right) - \frac{\left(\sigma(f,\Delta_n,x_k)\right)^p}{\left(\sigma(g,\Delta_n,x_k)\right)^{p-1}} - \frac{\sigma(f,\Delta_n,x_k)}{\sigma(g,\Delta_n,x_k)}\left(\sigma\left(\frac{f^{p-1}}{g^{p-2}}, \Delta_n, x_k\right) - \frac{\left(\sigma(f,\Delta_n,x_k)\right)^{p-1}}{\left(\sigma(g,\Delta_n,x_k)\right)^{p-2}}\right)\right)$$

and

$$0 \le \sigma\left(\frac{f^p}{g^{p-1}}, \Delta_n, x_k\right) - \frac{\left(\sigma(f,\Delta_n,x_k)\right)^p}{\left(\sigma(g,\Delta_n,x_k)\right)^{p-1}} \le \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})\sigma(g,\Delta_n,x_k).$$

We considered here $\sigma\left(\frac{f^p}{g^{p-1}}, \Delta_n, x_k\right)$ is the corresponding Riemann sum of function $\frac{f^p}{g^{p-1}}, \Delta_n = \left(x_0, x_1,..., x_n\right)$ division, and the intermediate x_k points. When *n* tends to infinity, in previous inequality the limits become:

$$0 \le \int_a^b \frac{\left(f(x)\right)^p}{\left(g(x)\right)^{p-1}} dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} \le$$

$$\le p\left[\int_a^b \frac{\left(f(x)\right)^p}{\left(g(x)\right)^{p-1}} dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} - \frac{\int_a^b f(x)dx}{\int_a^b g(x)dx}\left(\int_a^b \frac{\left(f(x)\right)^{p-1}}{\left(g(x)\right)^{p-2}} dx - \frac{\left(\int_a^b f(x)dx\right)^{p-1}}{\left(\int_a^b g(x)dx\right)^{p-2}}\right)\right]$$

and

$$0 \le \int_a^b \frac{\left(f(x)\right)^p}{\left(g(x)\right)^{p-1}} dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} \le \frac{p}{4}(M-m)(M^{p-1}-m^{p-1})\int_a^b g(x)dx.$$

The next result is the integral form of the inequality (2.19) of Theorem 2.9, from [7].

Theorem 6. If $p \geq 1$, f and g are two continuous functions $f, g:[a,b] \to R_+$ on [a,b], with $m = \inf_{x\in[a,b]} \frac{f(x)}{g(x)}$, $M = \sup_{x\in[a,b]} \frac{f(x)}{g(x)}$ then we have:

$$0 \leq \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} \leq$$

$$\leq \frac{\left[(M+m)\int_a^b g(x)dx - \int_a^b f(x)dx\right]^p}{\left(\int_a^b f(x)dx\right)^{p-1}} - \frac{(M+m)^p}{2^{p-1}}\int_a^b g(x)dx + \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx.$$

Proof: We will use the same techniques as in previous proof, choosing $x_k = k + \frac{b-a}{n}, k \in \{0,1,...,n\}$, using Theorem 2.9, Riemann sum of the corresponding functions, $\Delta_n = (x_0, x_1, ..., x_n)$ division, and the intermediate x_k points. Then when n tends to infinity, the limits obtained form the inequality from theorem.

The following integral inequality results from Theorem 3.

Consequence 1. If $p \geq 1$, and f and g are two continuous functions $f, g:[a,b] \to R_+$ on [a,b], with $g(x) > 0$, where $m = \inf_{x\in[a,b]} \frac{f(x)}{g(x)}$, $M = \sup_{x\in[a,b]} \frac{f(x)}{g(x)}$ then we have:

$$0 \leq \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} \leq \left[M^p + m^p - \frac{(M+m)^p}{2^{p-1}}\right]\int_a^b g(x)dx.$$

We will give now the integral form of the inequality (2.13), Theorem 2.5, see [7].

Theorem 7. Let $f, g:[a,b] \to R_+$ be two integrable functions on [a, b] with $g(x) > 0, (\forall) x \in [a,b], p > 1$ and $mg(x) \leq f(x) \leq Mg(x), (\forall) x \in [a,b]$. Then we have the inequality:

$$\frac{p(p-1)m^{p-2}}{\int_a^b g(x)dx}\int_a^b\int_a^b \frac{(f(x)g(y) - f(y)g(x))^2}{g(x)g(y)} dxdy \leq \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}} dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} \leq$$

$$\leq \frac{p(p-1)M^{p-2}}{\int_a^b g(x)dx}\int_a^b\int_a^b \frac{(f(x)g(y) - f(y)g(x))^2}{g(x)g(y)} dxdy.$$

Proof: Using the definition of double integral and taking

$$x_k = k + \frac{b-a}{n}, y_j = j + \frac{b-a}{n}, k \in \{0,1,\dots,n\}, j \in \{0,1,\dots,m\}$$

we have

$$\int_a^b\int_a^b \frac{\left(f(x)g(y) - f(y)g(x)\right)^2}{g(x)g(y)} dxdy = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\left(f(x_i)g(y_j) - f(y_j)g(x_i)\right)^2}{g(x_i)g(y_j)} (x_{i-1} - x_i)(y_{j-1} - y_j).$$

When $n = m$ tends to infinity

$$\begin{aligned}
&\int_a^b\int_a^b \frac{\left(f(x)g(y) - f(y)g(x)\right)^2}{g(x)g(y)} dxdy = \\
&= 2\lim_{n,m\to\infty} \sum_{1\le i<j\le n} \frac{\left(f(x_i)g(y_j) - f(y_j)g(x_i)\right)^2}{g(x_i)g(y_j)} (x_{i-1} - x_i)(y_{j-1} - y_j) = \\
&= 2\lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{\left(f(x_i)g(x_j) - f(x_j)g(x_i)\right)^2}{g(x_i)g(x_j)} (x_{i-1} - x_i)(x_{j-1} - x_j) = \\
&= \lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{\left(f(x_i)g(x_j) - f(x_j)g(x_i)\right)^2}{g(x_i)g(y_j)} \frac{(b-a)^2}{n^2}.
\end{aligned}$$

and using Theorem 2.5,

$$\begin{aligned}
&\frac{p(p-1)m^{p-1}}{\sum_{i=1}^{n} g(x_i)\frac{(b-a)}{n}} \sum_{1\le i<j\le n} \frac{(f(x_i)g(x_j) - f(x_j)g(x_i))^2}{g(x_i)g(x_j)} \frac{(b-a)^2}{n^2} \le \\
&\le \sum_{i=1}^{n} \frac{(f(x_i))^p}{(g(x_i))^{p-1}} \frac{b-a}{n} - \frac{\left(\sum_{i=1}^{n} f(x_i)\frac{b-a}{n}\right)^p}{\left(\sum_{i=1}^{n} g(x_i)\frac{b-a}{n}\right)^{p-1}} \le \\
&\le \frac{p(p-1)M^{p-1}}{\sum_{i=1}^{n} g(x_i)\frac{(b-a)}{n}} \sum_{1\le i<j\le n} \frac{(f(x_i)g(x_j) - f(x_j)g(x_i))^2}{g(x_i)g(x_j)} \frac{(b-a)^2}{n^2}
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{p(p-1)m^{p-2}}{\int_a^b g(x)dx} \int_a^b\int_a^b \frac{\left(f(x)g(y) - f(y)g(x)\right)^2}{g(x)g(y)} dxdy \le \int_a^b \frac{\left(f(x)\right)^p}{\left(g(x)\right)^{p-1}} dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} \le \\
&\le \frac{p(p-1)M^{p-2}}{\int_a^b g(x)dx} \int_a^b\int_a^b \frac{\left(f(x)g(y) - f(y)g(x)\right)^2}{g(x)g(y)} dxdy,
\end{aligned}$$

that is the inequality from theorem.

If we compute the double integral from previous theorem we deduce the following inequality:

Consequence 2. Let $f, g : [a,b] \to R_+$ two integrable functions on [a,b], with $g(x) > 0$, $(\forall) x \in [a,b]$, $p > 1$ and $mg(x) \le f(x) \le Mg(x), (\forall) x \in [a,b]$. Then we have the following inequality:

$$p(p-1)m^{p-2}\left(\int_a^b \frac{f^2(x)}{g(x)}dx - \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b g(x)dx}\right) \le \int_a^b \frac{(f(x))^p}{(g(x))^{p-1}}dx - \frac{\left(\int_a^b f(x)dx\right)^p}{\left(\int_a^b g(x)dx\right)^{p-1}} \le$$

$$\le p(p-1)M^{p-2}\left(\int_a^b \frac{f^2(x)}{g(x)}dx - \frac{\left(\int_a^b f(x)dx\right)^2}{\int_a^b g(x)dx}\right).$$

Using from [5], the inequality,

$$\sum_{k=1}^n \frac{x_k^{p+1}}{a_k^p} \le \frac{\left(\sum_{k=1}^n x_k\right)^{p+1}}{\left(\sum_{k=1}^n a_k\right)^p}, \; p \in (-1,0)$$

which is the reverse inequality of (1), and the same techniques as in Theorem 4 we obtain below the integral form of previous inequality:

Remark 1. If $a, b \in R, a < b, p \in (-1,0)$, $f, g : [a,b] \to [0,\infty)$ are integrable functions on [a,b], $g(x) \ne 0$ for any $x \in [a,b]$, then

$$\int_a^b \frac{(f(x))^{p+1}}{(g(x))^p}dx \le \frac{\left(\int_a^b f(x)dx\right)^{p+1}}{\left(\int_a^b g(x)dx\right)^p}.$$

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