*ORIGINAL PAPER*

# POISSON SUMMATION FORMULA ASSOCIATED WITH THE FRACTIONAL LAPLACE TRANSFORM

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***Abstract.*** *The linear canonical transform is four parameterized integral transform, which is an important tool in signal processing and optics. The application of linear canonical transform in quantum mechanics has focused attention on its complex extension. Fractional Laplace transform is a special case of complex linear canonical transform. The present paper investigates the generalized Poisson's summation formula for the Laplace domain and used it to derive Poisson's summation formula for the fractional Laplace transform of the periodic functions of compact support. Then some new results associated with this novel formula have been presented.*

***Keywords:*** *Linear canonical transform, Poisson summation formula, Fractional Fourier transform, Fractional Laplace transform.*

***AMS Code:*** *44A10, 46F12*

## 1. INTRODUCTION

Linear canonical transform introduced by Moshinsky [4] in 1971, has received much attention in recent years as a generalization of fractional Laplace transform. It is prominently used in optics [7], radar system analysis [8], signal processing [6] etc. Many transforms such as Fourier transform, fractional Fourier transform, Fresnel transform, Chirp transform are all special cases of linear canonical transform. Many of its properties are found in the literature. Eigen values of linear canonical transform in [6], Convolution and product theorem in [9], Uncertainty principal in [7]. The properties and applications of the sampling formulae in the traditional Fourier domain have been studied and sampling signal analysis have been extended for band limited signals in fractional Fourier domain in the literature. A generalized Poisson summation formula and its applications to fast linear convolution have been studied in [5].

Actually Kramer suggested in 1972 that some problem in nuclear duster theory may be solved by the complexification of the parameters of linear canonical transform [10]. Indeed Torre [8] had given proper definition of fractional Laplace transform as a special case of complex linear canonical transform. Analytical study of fractional Laplace transform can be seen in [1], where as convolution of two version of fractional Laplace transform in [2]. In Fourier analysis Poisson summation formula is the relation that defines the periodic summation of a function in terms of a Fourier transform of discrete samples of the original function. The Poisson sum formulae associated with the fractional Fourier transform was

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studied in [3]. The objective of this paper is to study the Poisson summation formula in fractional Laplace transform domain.

This paper is organized as follows; in section 2 the preliminaries are presented. In section 3, the Poisson summation formula in Laplace domain is obtained, which is used to get Poisson summation formula in fractional Laplace transform domain. The results based on Poisson summation formula have been presented.. Some important properties of Poisson summation formula are given in section 4. Finally the conclusion and future working directions are given in section 5.

## 2. PRELIMINARIES

### *2.1 FRACTIONAL LAPLACE TRANSFORM*

In full analogy with fractional Fourier transform, Torre [8] introduced fractional Laplace transform as,

$$L^{\alpha}(u)=\int_{-\infty}^{\infty} f(t)K_{\alpha}(t,u)dt$$

where $f(t)$ is any square integrable function and

$$K_{\alpha}(t,u)=\begin{cases}\sqrt{\dfrac{1-i\cot\alpha}{2\pi i}}e^{\frac{t^2}{2}\cot\alpha+\frac{u^2}{2}\cot\alpha-tu\operatorname{cosec}\alpha} & ,\ \alpha \textit{ is not multiple of } \pi \\ \delta(t-u) & ,\ \alpha \textit{ is a multiple of } \pi\end{cases} \tag{1}$$

This paper deals with the case when $\alpha \neq k\pi$. Fractional Laplace transform defined in (1) is the generalization of classical Laplace transform.

### *2.2. POISSON SUMMATION FORMULA IN THE FOURIER DOMAIN*

The Poisson sum formula demonstrates that the sum of infinite samples in time domain of a function $f(t)$ is equivalent to the sum of infinite samples of $F(w)$ in Fourier domain. This can be represented as follows:

$$\sum_{k=-\infty}^{\infty} f(t+k\tau)=\frac{1}{\tau}\sum_{n=-\infty}^{\infty} F\left(\frac{n}{\tau}\right)e^{\frac{\text{int}}{\tau}}$$

or

$$\sum_{k=-\infty}^{\infty} f(k\tau)=\frac{1}{\tau}\sum_{n=-\infty}^{\infty} F\left(\frac{n}{\tau}\right)$$

where $F(w)$ is the Fourier transform of the function $f(t)$.

*2.3 FUNCTIONS WITH COMPACT SUPPORT IN FOURIER DOMAIN*

Let $F(w)$ be the Fourier transform of the function $f(t)$. Then $f(t)$ is said to have compact support in Fourier domain if $F(w)=0$ for

$$|w| > \Omega \tag{2}$$

where $\Omega$ is some real number.

*2.4 GAUSSIAN PERIODIC FUNCTION*

We call a function $f(t)$ as a Gaussian periodic with period $\tau$ and order $\alpha$ if,

$$e^{\frac{t^2}{2}\cot\alpha} f(t) = e^{\frac{(t+\tau)^2}{2}\cot\alpha} f(t+\tau) \tag{3}$$

## 3. GENERALIZED POISSON SUMMATION FORMULAE

Using the concept of generalized Fourier transform here we obtained the Poisson summation formula for Laplace transform.

*3.1 POISSON SUMMATION FORMULAS FOR LAPLACE TRANSFORM*

By the definition of Laplace transform

$$L\{f(t)\}(s) = \int_{-\infty}^{\infty} f(t)e^{-st}dt \text{, where } s \in C$$

Let $s = \sigma + iw$ then

$$L\{f(t)\}(s) = F\{g(t)\}(w) = G(w) \tag{4}$$

where $g(t) = f(t)e^{-\sigma\tau}$

Here we should note that $\sigma$ is real, hence it cannot be expressed as a Fourier transform on all $L^2(R)$ but as in [10] this can be done only when Laplace transform of $f(t)$ has compact support. Thus (4) can be defined for the function of compact support. We assume that $f(t)$ is the function such that its Laplace transform has compact support say $\Omega_\alpha$, that is $|u| > \Omega_\alpha$ then $L\{f(t)\}(s) = 0$. Now we apply Poisson's formula for Fourier transform and write,

$$\sum_{-\infty}^{\infty} g\left(t+k\tau\right)=\frac{1}{\tau}\sum_{-\infty}^{\infty} G\left(\frac{n}{\tau}\right)e^{\frac{\text{int}}{\tau}}$$

$$\sum_{-\infty}^{\infty} e^{-\sigma\left(t+k\tau\right)} f\left(t+k\tau\right)=\frac{1}{\tau}\sum_{-\infty}^{\infty} L\left\{f\left(t\right)\right\}\left(\frac{n}{\tau}\right)e^{\frac{\text{int}}{\tau}} \tag{5}$$

$$\sum_{-\infty}^{\infty} e^{-\sigma k\tau} f\left(t+k\tau\right)=\frac{e^{\sigma\tau}}{\tau}\sum_{-\infty}^{\infty} L\left\{f\left(t\right)\right\}\left(\frac{n}{\tau}\right)e^{\frac{\text{int}}{\tau}}$$

### *3.2 POISSON SUMMATION FORMULAS FOR FRACTIONAL LAPLACE TRANSFORM*

Let $f\left(t\right)$ be the function having compact support in the fractional Laplace domain that is if $\left[\left[L\right]^{\alpha} f\left(t\right)\right]=L^{\alpha}\left(u\right)$ for $\left|u\right|>\Omega_{\alpha}$, $L^{\alpha}\left(u\right)=0$

By the definition of fractional Laplace transform, we have

$$L^{\alpha}\left(u\right)=\sqrt{\frac{1-i\cot\alpha}{2\pi i}}\int_{-\infty}^{\infty} f\left(t\right)e^{\frac{t^2}{2}\cot\alpha+\frac{u^2}{2}\cot\alpha-tu\csc\alpha}dt$$

$$\sqrt{\frac{2\pi i}{1-i\cot\alpha}}e^{-\frac{u^2}{2}\cot\alpha}L^{\alpha}\left(u\right)=\int_{-\infty}^{\infty}\left\{f\left(t\right)e^{\frac{t^2}{2}\cot\alpha}\right\}e^{-\left(u.\csc\alpha\right)t}dt=$$

$$=L\left\{g\left(t\right)\right\}\left(u.\csc\alpha\right)=\left(V\right)$$

where

$$g\left(t\right)=f\left(t\right)e^{\frac{t^2}{2}\cot\alpha} \tag{6}$$

$$\therefore G\left(V\right)=e^{-\frac{u^2}{2}\cot\alpha}L^{\alpha}\left(u\right)\sqrt{\frac{i}{1-i\cot\alpha}}=\sqrt{\frac{i}{1-i\cot\alpha}}e^{-\frac{\left(V^2\sin^2\alpha\right)^2}{2}\cot\alpha}L^{\alpha}\left(V\sin\alpha\right)=L\left\{g\left(t\right)\right\}\left(V\right)$$

when $V=u.\csc\alpha$

Poisson formula for Laplace transform is

$$\sum_{k=-\infty}^{\infty} e^{-\sigma k\tau}g\left(t+k\tau\right)=\frac{e^{\sigma\tau}}{\tau}\sum_{n=-\infty}^{\infty} L\left\{g\left(t\right)\right\}\left(\frac{n}{\tau}\right)e^{\frac{\text{int}}{\tau}}=\frac{e^{\sigma\tau}}{\tau}\sum_{-\infty}^{\infty} G\left(\frac{n}{\tau}\right)$$

$$\sum_{k=-\infty}^{\infty} e^{-\sigma k\tau}e^{\frac{1}{2}\left(t+k\sigma\right)^2\cot\alpha} f\left(t+k\tau\right)=\frac{e^{\sigma\tau}}{\tau}\sum_{n=-\infty}^{\infty}\sqrt{\frac{i}{1-i\cot\alpha}}e^{-\frac{1}{2}\sin^2\alpha\cot\alpha\frac{n^2}{\tau^2}}L^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right)e^{\frac{\text{int}}{t}}$$

$$\sum_{k=-\infty}^{\infty} e^{-\sigma k\tau}e^{\frac{1}{2}\left(2tk\tau+k^2\tau^2\right)\cot\alpha} f\left(t+k\tau\right)=\frac{e^{\sigma\tau}}{\tau}C_{\alpha}\sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}\sin^2\alpha\cot\alpha\frac{n^2}{\tau^2}}L^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right)e^{\frac{\text{int}}{t}} \tag{7}$$

where $C_{\alpha}=\sqrt{\dfrac{i}{1-i\cot\alpha}}$

*3.3 POISSON SUMMATION FORMULAS FOR FRACTIONAL LAPLACE TRANSFORM WHEN t=0*

Put $t=0$, equation (7) becomes

$$\sum_{k=-\infty}^{\infty} e^{-\sigma k\tau} e^{\frac{1}{2}(k\tau)^2 \cot\alpha} f(k\tau) = \frac{C_\alpha}{\tau} \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}\sin^2\alpha \cot\alpha \frac{n^2}{\tau^2}} L^\alpha\left(\frac{n\sin\alpha}{\tau}\right) \tag{8}$$

Equations (7) and (8) can be seen as Poisson summation formula associated with fractional Laplace transform of order $\alpha$. It is clear from the above equations that the infinite sum of the periodic replica of function $f(t)$ is equal to infinite sum of its fractional Laplace transform.

Next we prove three corollaries which give reduce forms of Poisson summation formula for special cases of the variable in the fractional Laplace transform.

*3.4 COROLLARY*

If $f(t)$ is in $\Omega_\alpha$ having compact support in fractional Laplace transform domain and $\frac{\sin\alpha}{\tau} > \Omega_\alpha$ then

$$\sum_{k=-\infty}^{\infty} e^{-\sigma k\tau} f(t+k\tau) e^{\frac{1}{2}(2tk\tau+k^2\tau^2)\cot\alpha} = \frac{1}{\tau}\frac{e^{\sigma\tau}}{C_\alpha} e^{-\frac{t^2}{2}\cot\alpha} L^\alpha(0)$$

where $C_\alpha = \sqrt{\frac{i}{1-i\cot\alpha}}$

*Proof:* $f(t)$ is in $\Omega_\alpha$ having compact support in fractional Laplace transform domain and $\frac{\sin\alpha}{\tau} > \Omega_\alpha$, then $L^\alpha\left(\frac{n\sin\alpha}{\tau}\right) = 0$ when $n \neq 0$.

From equation (5)

$$\sum_{k=-\infty}^{\infty} e^{-\sigma k\tau} f(t+k\tau) e^{\frac{1}{2}(2tk\tau+k^2\tau^2)\cot\alpha} = \frac{1}{\tau} e^{-\frac{t^2}{2}\cot\alpha} \frac{e^{\sigma\tau}}{C_\alpha} \sum_{n=-\infty}^{\infty} e^{-\frac{\cot\alpha}{2}\left(\frac{n\sin\alpha}{\tau}\right)^2} L^\alpha\left(\frac{n\sin\alpha}{\tau}\right) e^{i\left(\frac{n}{t}\right)t}$$

Given $\frac{\sin\alpha}{\tau} > \Omega_\alpha \quad \therefore \left|\frac{n\sin\alpha}{\tau}\right| > |n\Omega_\alpha| > \Omega_\alpha$ for all $n$ from $-\infty$ to $\infty$ except $n=0$.

$\therefore$ Right hand side exists only for $n=0$ and hence,

$$\sum_{k=-\infty}^{\infty} e^{-\sigma k\tau} f(t+k\tau) e^{\frac{1}{2}(2tk\tau+k^2\tau^2)\cot\alpha} = \frac{1}{\tau} e^{-\frac{t^2}{2}\cot\alpha} \frac{e^{\sigma\tau}}{C_\alpha} L^\alpha(0)$$

*3.5. COROLLARY*

If $f(t)$ is in $\Omega_\alpha$ having compact support in fractional Laplace domain and $\frac{\Omega_\alpha}{2} < \frac{\sin\alpha}{\tau} < \Omega_\alpha$, the Poisson sum formula reduces to,

$$\sum_{k=-\infty}^{\infty} C_\alpha e^{-\sigma k\tau} f(t+k\tau) e^{\frac{1}{2}\left(2tk\tau+k^2\tau^2\right)\cot\alpha} =$$

$$= \frac{e^{\sigma\tau}}{\tau} e^{-\frac{\cot\alpha}{2}\tau^2} \left\{ L^\alpha(0) + e^{-\frac{\cot\alpha}{2}\left(\frac{\sin\alpha}{\tau}\right)^2} \left[ L^\alpha\left(\frac{\sin\alpha}{\tau}\right) e^{i\left(\frac{n}{\tau}\right)t} + L^\alpha\left(-\frac{\sin\alpha}{\tau}\right) e^{-i\left(\frac{n}{\tau}\right)t} \right] \right\}$$

*Proof:* Given that $\frac{\Omega_\alpha}{2} < \frac{\sin\alpha}{\tau} < \Omega_\alpha$

$$\therefore \left|\frac{\sin\alpha}{\tau}\right| > \left|\frac{\Omega_\alpha}{2}\right| \quad \text{and} \quad \left|\frac{\sin\alpha}{\tau}\right| < \Omega_\alpha$$

$\therefore$ Only for $n = 0,\ 1$ and $-1$, we get $\left|\frac{n\sin\alpha}{\tau}\right| < \Omega_\alpha$

Otherwise for all other values of *n*, $\left|\frac{n\sin\alpha}{\tau}\right| > \Omega_\alpha$.

$\therefore$ Right hand side summation will contain only three nonzero terms for $n = 0,\ 1$ and $-1$.

*3.6 COROLLARY*

If $f(t)$ is in $\Omega_\alpha$ having compact support in fractional Laplace domain and as $\frac{\Omega_\alpha}{n} < \frac{\sin\alpha}{\tau} < \frac{\Omega_\alpha}{n-1}$ then the Poisson sum formula reduces to,

$$\sum_{k=-\infty}^{\infty} C_\alpha f(t+k\tau) e^{\frac{1}{2}\left(2tk\tau+k^2\tau^2\right)\cot\alpha} =$$

$$= \frac{e^{\sigma\tau}}{\tau} e^{-\frac{\cot\alpha}{2}\tau^2} \left\{ L^\alpha(0) + \sum_{k=1}^{n} e^{-\frac{\cot\alpha}{2}\left(\frac{k\sin\alpha}{\tau}\right)^2} \left[ L^\alpha\left(\frac{k\sin\alpha}{\tau}\right) e^{i\left(\frac{kt}{\tau}\right)} + L^\alpha\left(-\frac{k\sin\alpha}{\tau}\right) e^{-i\left(\frac{kt}{\tau}\right)} \right] \right\}$$

*Proof:* It is clear when $\frac{\Omega_\alpha}{n} < \frac{\sin\alpha}{\tau} < \frac{\Omega_\alpha}{n-1}$, on right hand side only $(2n+1)$ (i.e. from –*n* to *n*) terms will have nonzero values, and all other terms will vanish, thus we get

$$\sum_{k\in Z} f(t+k\tau) e^{\frac{1}{2}\left(2tk\tau+k^2\tau^2\right)\cot\alpha} =$$

$$= \frac{e^{\sigma\tau}}{\tau} e^{-\frac{\cot\alpha}{2}\tau^2} \left\{ L^\alpha(0) + \sum_{k=1}^{n} e^{-\frac{\cot\alpha}{2}\left(\frac{k\sin\alpha}{\tau}\right)^2} \left[ L^\alpha\left(\frac{k\sin\alpha}{\tau}\right) e^{i\left(\frac{kt}{\tau}\right)} + L^\alpha\left(-\frac{k\sin\alpha}{\tau}\right) e^{-i\left(\frac{kt}{\tau}\right)} \right] \right\}$$

*3.7 THEOREM:*

Appling the similar results in corollary 3.4, 3.5 and 3.6 to the equation (8) we get the following result

$$\sum_{k=-\infty}^{\infty} e^{\frac{1}{2}\left(2tk\tau+k^2\tau^2\right)\cot\alpha} f\left(k\tau\right)=\left\{\square\left(C_{\downarrow}\alpha/\tau L^{\uparrow}\alpha\left(0\right),\sin\alpha/\tau>\Omega_{\downarrow}\left(\alpha\right)\right.\right.$$

## 4. PROPERTY

In order to device some properties of infinite sum of fractional Laplace transform which can be obtained from the function $\sum_{k=-\infty}^{\infty} e^{-\sigma k\tau} e^{\frac{1}{2}\left(2tk\tau+k^2\tau^2\right)\cot\alpha} f\left(t+k\tau\right)$, we treat this as a special function of *t*, say

$$y\left(t\right)=\sum_{k=-\infty}^{\infty} f\left(t+k\tau\right)e^{-\sigma k\tau}e^{\frac{1}{2}\left(2tk\tau+k^2\tau^2\right)\cot\alpha} \qquad (9)$$

Then $y\left(t\right)$ is Gaussian periodic function with period $\tau$.

*Proof:* Consider

$$y\left(t+\tau\right)e^{\frac{1}{2}\left(t+\tau\right)^2\cot\alpha}=\sum_{k=-\infty}^{\infty} f\left(t+\tau+k\tau\right)e^{\frac{1}{2}\left\{2t\left(k+1\right)\tau+\left(k+1\right)^2\tau^2\right\}\cot\alpha}e^{\frac{1}{2}t^2\cot\alpha}=$$

$$=y\left(t\right)e^{\frac{1}{2}t^2\cot\alpha}$$

Hence $y\left(t\right)$ is Gaussian periodic.

The above derived results on Poisson summation formula are applicable to the following:

**Example:** The function having compact support is mostly used in signal processing. The above result estimate the original function having compact support**.**

If $f\left(t\right)$ is known then for an arbitrary value of $\tau$, find out $y\left(t\right)$. $y\left(t\right)$ is the Gaussian periodic with *N* number of fractional Fourier series. Then using the property, $f\left(t\right)$ is a function with compact support in fractional Laplace transform domain of order $\alpha$. The compact support is then $\Omega_\alpha \le \frac{2\pi N}{\tau\csc\alpha}$.

If $y\left(t\right)$ satisfy the conditions similar to that of last corollary then $\Omega_\alpha$ of $f\left(t\right)$ is in the range $\frac{\left(n-1\right)\sin\alpha}{\tau}<\Omega_\alpha<\frac{n\sin\alpha}{\tau}$.

## 5. CONCLUSIONS

In this work, a generalized Poisson summation formula has been proposed. It is obtained on the basis of Laplace transform and fractional Laplace transform. This paper investigates the generalized form of Poisson sum formula in fractional Laplace transform domain. Some novel results associated with this summation formula have been derived in the form of corollaries.

We also propose to provide the sampling theorem for the periodic functions with compact support in the fractional Laplace transform domain so that whenever the function needs to reconstruct from its discrete form, it can be done.

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