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ORIGINAL PAPER

ON A CLASS OF HOLOMORPHIC FUNCTION

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Abstract. *In this paper, we study two subclasses* $V(\alpha,\lambda;g)$ *and* $W(\alpha,\lambda;g)$ *of holomorphicfunctions* $f(z)$ *in the open unit disk U associated with some holomorphic function* $g(z)$. *The object of the present paper is to drive some interesting conditions and necessary conditions for* $f(z)$ *belonging to the classes* $V(\alpha,\lambda;g)$ *and* $W(\alpha,\lambda;g)$ *and some results of Jack's Lemma [3].*

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1. INTRODUCTION AND PRELIMINARIES

Let A denote the class of functions analytic in the unit disk
$U = \{z \in U : |z| < 1\}$

of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in U) \tag{1}$$

Definition 1.1. We say that $f(z) \in V(\alpha,\lambda;g)$ if it satisfies

$$\left| zf''(z) - e^{i\alpha}\frac{g(z)}{z} \right| < \lambda \qquad (z \in U) \tag{2}$$

for some real $\alpha(-\pi \le \alpha \le \pi)$, $\lambda > 1$, and for some $g(z) \in A$.

Definition 1.2. We say that $f(z) \in W(\alpha,\lambda;g)$

$$\left| \frac{f(z)}{z} - ze^{i\alpha} g''(z) \right| < \lambda \qquad (z \in U) \tag{3}$$

for some real $\alpha(-\pi \le \alpha \le \pi)$, $\lambda > 1$, and for some $g(z) \in A$.

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2. SUFFICIENT CONDITIONS

To discuss our problem, we consider

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \qquad (z \in U) \tag{4}$$

with $f(z) \in A$ in (1) and $g(z) \in A$ in (4), we derive the following theorem.

Theorem 2.1. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \left| n(n-1)a_n - e^{i\alpha} b_n \right| \le \lambda - 1 \tag{5}$$

for some $\alpha(-\pi \le \alpha \le \pi)$, $\lambda > 1$, and for some $g(z) \in A$, then $f(z) \in V(\alpha, \lambda; g)$.

Proof: For $f(z)$ in (1) and $g(z)$ in (4), we have

$$\left| zf''(z) - e^{i\alpha} \frac{g(z)}{z} \right| = \left| z\left(\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} \right) - e^{i\alpha} \left(\frac{z + \sum_{n=2}^{\infty} b_n z^n}{z} \right) \right| =$$

$$= \left| \sum_{n=2}^{\infty} n(n-1) a_n z^{n-1} - e^{i\alpha} \left(1 + \sum_{n=2}^{\infty} b_n z^n \right) \right| \le$$

$$\le \sum_{n=2}^{\infty} \left| n(n-1) a_n - e^{i\alpha} b_n \right| |z|^{n-1} + \left| e^{i\alpha} \right| <$$

$$< \sum_{n=2}^{\infty} \left| n(n-1) a_n - e^{i\alpha} b_n \right| + 1 <$$

$$< \lambda - 1 + 1 = \lambda$$

Therefore, $f(z) \in V(\alpha, \lambda; g)$.

Example 2.2. For any function $g(z) \in A$, let us define function $f(z) \in A$ with

$$a_n = \frac{1}{n(n-1)} \left(\frac{\lambda - 1}{n(n-1)} e^{i\gamma} + e^{i\alpha} b_n \right) \qquad (n = 2,\ 3,\ 4,\ ...) \tag{6}$$

Then, clearly we have

$$\sum_{n=2}^{\infty} \left| n(n-1) a_n - e^{i\alpha} b_n \right| = \sum_{n=2}^{\infty} \left| \frac{\lambda - 1}{n(n-1)} e^{i\gamma} + e^{i\alpha} b_n - e^{i\alpha} b_n \right| = (\lambda - 1) \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \left| e^{i\gamma} \right| =$$

$$= (\lambda - 1) \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = (\lambda - 1) \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \lambda - 1$$

which shows that $f(z) \in V(\alpha, \lambda; g)$.

Corollary 2.3. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \left| n(n-1)|a_n| - |b_n| \right| \leq \lambda - 1 \tag{7}$$

for some $\alpha(-\pi \leq \alpha \leq \pi)$, $\lambda > 1$, and for some $g(z) \in A$, with

$$\arg(a_n) - \arg(b_n) = \alpha \qquad (n = 2,\ 3,\ 4,\ ...) \tag{8}$$

then $f(z) \in V(\alpha, \lambda; g)$.

Proof: Since $\arg(a_n) - \arg(b_n) = \alpha$, we can write

$$\arg(a_n) = \varphi_n, \quad \arg(b_n) = \varphi_n - \alpha \qquad n = 2,\ 3,\ 4,\ ... \tag{9}$$

This implies that

$$\begin{aligned} &\left| n(n-1)a_n - e^{i\alpha} b_n \right| = \left| n(n-1) \right| a_n \left| e^{i\varphi_n} - e^{i\alpha} \right| b_n \left| e^{i(\varphi_n - \alpha)} \right| = \\ &= \left| n(n-1) \right| a_n \left| e^{i\varphi_n} - |b_n| e^{i\varphi_n} \right| = \left| \left(n(n-1)|a_n| - |b_n| \right) e^{i\varphi_n} \right| = \\ &= \left| n(n-1)|a_n| - |b_n| \right| \left| e^{i\varphi_n} \right| = \left| n(n-1)|a_n| - |b_n| \right| \end{aligned}$$

so that $f(z) \in V(\alpha, \lambda; g)$.

Corollary 2.4. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \sqrt{n^2(n-1)^2 - 2n(n-1)\cos\alpha + 1}\,|a_n| \leq \lambda - 1 \tag{10}$$

for some $\alpha(-\pi \leq \alpha \leq \pi)$, $\lambda > 1$, then $f(z) \in V(\alpha, \lambda; f)$.

Applying the same method for $f(z) \in W(\alpha, \lambda; g)$, we have

Theorem 2.5. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \left| a_n - e^{i\alpha} n(n-1) b_n \right| \leq \lambda - 1 \tag{11}$$

for some $\alpha(-\pi \leq \alpha \leq \pi)$, $\lambda > 1$, and for some $g(z) \in A$, then $f(z) \in W(\alpha, \lambda; g)$.

Example 2.6. For any function $g(z) \in A$, let us define function $f(z) \in A$ with

$$a_n = \frac{\lambda - 1}{n(n-1)} e^{i\gamma} + e^{i\alpha} n(n-1) b_n \qquad (n = 2,\ 3,\ 4,\ ...) \tag{12}$$

Then, clearly we have

$$\sum_{n=2}^{\infty} \left| a_n - e^{i\alpha} n(n-1) b_n \right| = \sum_{n=2}^{\infty} \left| \frac{\lambda - 1}{n(n-1)} e^{i\gamma} + e^{i\alpha} n(n-1) b_n - e^{i\alpha} n(n-1) b_n \right| =$$
$$= (\lambda - 1) \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \left| e^{i\gamma} \right| = (\lambda - 1) \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = (\lambda - 1) \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) = \lambda - 1$$

which shows that $f(z) \in W(\alpha, \lambda; g)$.

Corollary 2.7. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \left| |a_n| - n(n-1)|b_n| \right| \le \lambda - 1 \tag{13}$$

for some $\alpha(-\pi \le \alpha \le \pi)$, $\lambda > 1$, and for some $g(z) \in A$, with

$$\arg(a_n) - \arg(b_n) = \alpha \qquad (n = 2,\ 3,\ 4,\ ...) \tag{14}$$

then $f(z) \in W(\alpha, \lambda; g)$.

Corollary 2.8. If $f(z) \in A$ satisfies

$$\sum_{n=2}^{\infty} \sqrt{n^2(n-1)^2 - 2n(n-1)\cos\alpha + 1}\, |a_n| \le \lambda - 1 \tag{15}$$

for some $\alpha(-\pi \le \alpha \le \pi)$, $\lambda > 1$, then $f(z) \in W(\alpha, \lambda; f)$.

3. NECESSARY CONDITIONS

Next, we discuss some necessary conditions for $f(z)$ belonging to $V(\alpha, \lambda; g)$ and $W(\alpha, \lambda; g)$.

Theorem 3.1. If $f(z) \in V(\alpha, \lambda; g)$ with

$$\arg\left(n(n-1) a_n - e^{i\alpha} b_n\right) = (n-1)\varphi \qquad (n = 2,\ 3,\ 4,\ ...)$$

then

$$\sum_{n=2}^{\infty}\left|n(n-1)a_n - e^{i\alpha}b_n\right| \le \lambda + \cos\alpha \tag{16}$$

Proof: Assume that $f(z)\in V(\alpha,\lambda;g)$, then we see that

$$\left|zf''(z)-e^{i\alpha}\frac{g(z)}{z}\right| = \left|\sum_{n=2}^{\infty}n(n-1)a_n z^{n-1} - e^{i\alpha}\left(1+\sum_{n=2}^{\infty}b_n z^{n-1}\right)\right| =$$

$$= \left|\sum_{n=2}^{\infty}\left(n(n-1)a_n - e^{i\alpha}b_n\right)z^{n-1} - e^{i\alpha}\right| = \left|\sum_{n=2}^{\infty}\left|n(n-1)a_n - e^{i\alpha}b_n\right|e^{i(n-1)\varphi}z^{n-1} - e^{i\alpha}\right| < \lambda$$

for all $z\in U$. We take a point $z\in U$ such that $\arg(z)=-\varphi$. Then, we have $z^{n-1}=|z|^{n-1}e^{-i(n-1)\varphi}$. Therefore

$$\left|zf''(z)-e^{i\alpha}\frac{g(z)}{z}\right| = \left|\sum_{n=2}^{\infty}\left|n(n-1)a_n - e^{i\alpha}b_n\right|\left|z^{n-1}\right| - e^{i\alpha}\right|$$

for all $z\in U$. This implies that

$$\sum_{n=2}^{\infty}\left|n(n-1)a_n - e^{i\alpha}b_n\right|\left|z^{n-1}\right| - \cos\alpha < \lambda$$

for all $z\in U$. Therefore, letting $|z|\to 1^-$, we obtain

$$\sum_{n=2}^{\infty}\left|n(n-1)a_n - e^{i\alpha}b_n\right| \le \lambda + \cos\alpha$$ □

Taking $\alpha=\frac{\pi}{2}$ in Theorem 3.1, we have

Corollary 3.2. If $f(z)\in V(\alpha,\lambda;g)$ with

$$\arg\left(n(n-1)a_n - ib_n\right) = (n-1)\varphi \qquad (n=2,\ 3,\ 4,\ ...)$$

then

$$\sum_{n=2}^{\infty}\left|n(n-1)a_n - ib_n\right| \le \lambda \tag{17}$$

For the class $W(\alpha,\lambda;g)$, we have the following theorem.

Theorem 3.3. If $f(z) \in W(\alpha, \lambda; g)$ with

$$\arg\left(a_n - e^{i\alpha} n(n-1) b_n\right) = (n-1)\varphi \qquad (n = 2,\ 3,\ 4,\ \ldots)$$

then

$$\sum_{n=2}^{\infty} \left|a_n - e^{i\alpha} n(n-1) b_n\right| \le \lambda + \cos\alpha$$

Corollary 3.4. If $f(z) \in W\left(\frac{\pi}{2}, \lambda; g\right)$ with

$$\arg\left(a_n - in(n-1) b_n\right) = (n-1)\varphi \qquad (n = 2,\ 3,\ 4,\ \ldots)$$

then

$$\sum_{n=2}^{\infty} \left|a_n - in(n-1) b_n\right| \le \lambda$$

4. SOME RESULT OF JACK'S LEMMA

To discuss some result of Jack's lemma for $f(z)$, we recall here the following lemma due to Jack [1] or due to Miller and Mocanu [2], you can see also [4 - 6].

Lemma 4.1. Let $w(z)$ be holomorphic in U with $w(0) = 0$. If there exists a point $z_0 \in U$ such that

$$\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| \tag{18}$$

then $z_0 w'(0) = kw(0)$, where k is real and $k \ge 1$.

Theorem 4.2. If $f(z) \in A$ satisfies

$$\left|\frac{f(z)}{z} - ze^{i\alpha} g''(z)\right| < 2\lambda - 1 \qquad (z \in U) \tag{19}$$

for some $\alpha(-\pi \le \alpha \le \pi)$, $\lambda > \frac{1}{2}$, and for some $g(z) \in A$, then

$$\left|\frac{1}{z}\int_0^z \frac{f(t)}{t} dt - e^{i\alpha} \int_0^z t g''(t) dt\right| < \lambda + 1 \qquad (z \in U) \tag{20}$$

Proof: Let us define the function $w(z)$ in *U* by

$$\frac{1}{z}\int_0^z \frac{f(t)}{t}dt - e^{i\alpha}\int_0^z tg''(t)dt - 1 = \lambda w(z) \qquad (21)$$

Then, clearly, $w(0)=0$ and $w(z)$ is holomorphic in *U*. We want to prove that $w(z)$ satisfies $|w(z)|<1$ in *U*. By differentiating both sides of (21) and do some calculation we obtain

$$\frac{f(z)}{z} - ze^{i\alpha}g''(z) = 1 + \lambda w(z)\left(1 + \frac{zw'(z)}{w(z)}\right)$$

and hence

$$\left|\frac{f(z)}{z} - ze^{i\alpha}g''(z)\right| = \left|1 + \lambda w(z)\left(1 + \frac{zw'(z)}{w(z)}\right)\right| < 2\lambda - 1$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\le|z_0|}|w(z)| = |w(z_0)| = 1$$

Applying Jack's lemma to $w(z)$ at a point z_0, we can write

$$w(z_0) = e^{i\theta},\ z_0 w'(z_0) = kw(z_0) \qquad (k \ge 1).$$

This gives us that

$$\left|\frac{f(z_0)}{z_0} - e^{i\alpha}z_0 g''(z_0)\right| = \left|1 + \lambda e^{i\theta}(1+k)\right| \ge$$
$$\ge |\lambda(1+k)| - 1 < 2\lambda - 1$$

which contradicts our condition of the theorem. Thus, there is no point z_0 in *U* such that $|w(z_0)|=1$. This implies that $|w(z)|<1$ for all $z \in U$. Therefore, we have

$$\left|\frac{1}{z}\int_0^z \frac{f(t)}{t}dt - e^{i\alpha}\frac{1}{z}\int_0^z tg''(t)dt\right| < \lambda + 1 \qquad (z \in U)$$ □

Corollary 4. 3. If $f(z) \in A$ satisfies

$$\left|\frac{f(z)}{z} - izg''(z)\right| < 2\lambda - 1 \qquad (z \in U)$$

for some $\lambda > \frac{1}{2}$, then

$$\left| \frac{1}{z}\int_0^z \frac{f(t)}{t}dt - \frac{i}{z}\int_0^z tg''(t)dt \right| < \lambda + 1 \qquad (z \in U)$$

Corollary 4.4. If $f(z) \in A$ satisfies

$$\left| zf''(z) - e^{i\alpha}\frac{g(z)}{z} \right| < 2\lambda - 1 \qquad (z \in U)$$

for some real $\alpha(-\pi \le \alpha \le \pi)$, $\lambda > 1$, and for some $g(z) \in A$, for some $g(z) \in A$, then

$$\left| \frac{1}{z}\int_0^z tf''(t)dt - e^{i\alpha}\frac{1}{z}\int_0^z \frac{g(t)}{t}dt \right| < \lambda + 1 \qquad (z \in U) \tag{22}$$

Theorem 4.5. If $f(z) \in A$ satisfies

$$\mathrm{Re}\left(\frac{f(z)}{z} - e^{i\alpha} zg''(z) \right) > 1 - \frac{3}{4}\lambda \qquad (z \in U) \tag{23}$$

for some real $\alpha(-\pi \le \alpha \le \pi)$, $\lambda > \frac{1}{2}$, and for some $g(z) \in A$, then

$$\mathrm{Re}\left(\frac{1}{z}\int_0^z \frac{f(t)}{t}dt - e^{i\alpha}\int_0^z tg''(t)dt \right) > 1 - \frac{\lambda}{2} \qquad (z \in U) \tag{24}$$

Proof: Let us define the function $w(z)$ by

$$\frac{1}{z}\int_0^z \frac{f(t)}{t}dt - e^{i\alpha}\frac{1}{z}\int_0^z tg''(t)dt - 1 = \lambda\frac{w(z)}{1 - w(z)} \qquad (w(z) \ne 1)$$

Then, clearly, $w(0) = 0$ and $w(z)$ is holomorphic in *U*. By the definition for $w(z)$, we have that

$$\frac{1}{z}\int_0^z \frac{f(t)}{t}dt - e^{i\alpha}\int_0^z tg''(t)dt = \lambda\frac{w(z)}{1 - w(z)} + z \tag{25}$$

Differentiating both sides of (25) and do some calculation, we obtain

$$\frac{f(t)}{t} - e^{i\alpha} z g''(z) = 1 + \lambda \frac{w(z)}{1-w(z)} + \lambda \frac{zw'(z)}{(1-w(z))^2}$$

and hence

$$\operatorname{Re}\left(\frac{f(t)}{t} - e^{i\alpha} z g''(z) \right) = \operatorname{Re}\left(1 + \lambda \frac{w(z)}{1-w(z)} + \lambda \frac{zw'(z)}{(1-w(z))^2} \right) > 1 - \frac{3}{4}\lambda$$

for all $z \in U$. If there exists a point $z_0 \in U$ such that

$$\max_{|z| \le |z_0|} |w(z)| = |w(z_0)| = 1,$$

then by applying Jack's lemma at the point z_0, we can write

$$w(z_0) = e^{i\theta},\ z_0 w'(z_0) = k w(z_0) \qquad (k \ge 1).$$

This gives us that

$$\operatorname{Re}\left(\frac{f(z)}{z} - e^{i\alpha} z g''(z) \right) = 1 - \frac{\lambda}{2}\left(1 + \frac{k}{2}\right) \le 1 - \frac{3}{4}\lambda$$

which contradicts our condition. Therefore, $|w(z)| < 1$ for all $z \in U$ so that

$$\operatorname{Re}\left(\frac{w(z)}{1-w(z)} \right) > \frac{1^-}{2} \qquad (z \in U),$$

and hence

$$\operatorname{Re}\left(\frac{1}{z}\int_0^z \frac{f(t)}{t} dt - e^{i\alpha} \int_0^z t g''(t) dt \right) > 1 - \frac{\lambda}{2} \qquad (z \in U)$$ □

Corollary 4.6. If $f(z) \in A$ satisfies

$$\operatorname{Re}\left(z f''(z) - e^{i\alpha} \frac{g(z)}{z} \right) > -\cos\alpha - \frac{3}{4}\lambda \qquad (z \in U) \tag{26}$$

for some real $\alpha(-\pi \le \alpha \le \pi)$, $\lambda > 0$, and for some $g(z) \in A$, then

$$\mathrm{Re}\left(\frac{1}{z}\int_0^z tf''(t)\,dt - e^{i\alpha}\int_0^z \frac{g(t)}{t}\,dt + e^{i\alpha}\right) > -\cos\alpha - \frac{\lambda}{2} \qquad (z \in U) \tag{27}$$

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REFERENCES

[1] Jake, I.S., *J. London Math. Soc*., **3**, 469, 1971.
[2] Miller, S.S., Mocanu, P.T., *J. Math. Anal. Appl*., **65**, 289, 1978.
[3] Ali Abubaker, A.A., Darus, M., *Revista Notas de Matematica*, **7**(2), 119, 2011.
[4] Shiraishi, H., Owa, S., *Int. J. Open Problems Compt. Math*., **2**(1), 37, 2009.
[5] Ochiai, K., Owa, S., Acu, M., *J. General Mathematics*, **13**(2), 73, 2005.
[6] Shiraishi, H., Owa, S., *J. Studia Univ."Babes-Bolyai", Mathematica*, **LV**(3), 27, 2010.