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ORIGINAL PAPER

SOME GENERALIZATIONS OF IMO INEQUALITY

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Abstract. *In this paper we give some generalizations of a possible IMO inequality and some applications in triangles, quadrilaterals and tetrahedrons.*

Keywords: *Jensen's inequality, Radon's inequality, geometric inequalities.*

1. INTRODUCTION

At XXVIII-a, IMO, Havana (Cuba), 5-16 July 1987 was discussed in the jury the following problem, proposed by Greece:

Prove that in any triangle ABC*, with usual notations holds the inequality*:

$$\frac{a^n}{b+c}+\frac{b^n}{c+a}+\frac{c^n}{a+b}\geq\left(\frac{2}{3}\right)^{n-2}\cdot p^{n-1},\ \forall n\in N^* \tag{1}$$

We propose to generalize this problem and also to give some applications.

2. MAIN RESULTS

Theorem 1. *If* $n\in N^*-\{1\}, a,b,x_k\in R_+^*, k=\overline{1,n},\ X_n=\sum_{k=1}^{n}x_k$ *and* $m,t,u\in[1,\infty)$*, such that* $aX_n^t > b\max_{1\leq k\leq n} x_k^t$*, then*:

$$\sum_{k=1}^{n}\frac{x_k^m}{\left(aX_n^t-bx_k^t\right)^u}\geq\frac{n^{-m+tu+1}}{\left(an^t-b\right)^u}X_n^{m-tu} \tag{IMO}$$

Proof. Without loss of generality (WLOG) we can assume that:

$x_1\leq x_2\leq ...\leq x_n$, and then $x_1^m\leq x_2^m\leq ...\leq x_n^m$, thus

$aX_n^t-bx_1^t\geq aX_n^t-bx_2^t\geq ...\geq aX_n^t-bx_n^t \Leftrightarrow$

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$$\Leftrightarrow \frac{1}{\left(aX_n^t - bx_1^t\right)^u} \le \frac{1}{\left(aX_n^t - bx_2^t\right)^u} \le \ldots \le \frac{1}{\left(aX_n^t - bx_n^t\right)^u}.$$

By *P.L. Chebyshev*'s inequality we have that:

$$\sum_{k=1}^{n} \frac{x_k^m}{\left(aX_n^t - bx_k^t\right)^u} \ge \frac{1}{n} \cdot \left(\sum_{k=1}^{n} x_k^m\right) \cdot \sum_{k=1}^{n} \frac{1}{\left(aX_n^t - bx_k^t\right)^u} \tag{2}$$

Since the function $f : R_+^* \to R_+^*,\ f(x) = x^v$ with $v \in [1,\infty)$ is convex on R_+^*, by *Jensen*'s inequality we deduce that:

$$\sum_{k=1}^{n} x_k^v \ge n \cdot \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)^v = \frac{X_n^v}{n^{v-1}} \tag{3}$$

By *J. Radon*'s inequality we obtain that:

$$\sum_{k=1}^{n} \frac{1}{\left(aX_n^t - bx_k^t\right)^u} \ge \frac{n^{u+1}}{\left(\sum_{k=1}^{n}\left(aX_n^t - bx_k^t\right)\right)^u} = \frac{n^{u+1}}{\left(anX_n^t - b\sum_{k=1}^{n} x_k^t\right)^u} \tag{4}$$

If we taking account by (3), then (4) becomes:

$$\sum_{k=1}^{n} \frac{1}{\left(aX_n^t - bx_k^t\right)^u} \ge \frac{n^{u+1}}{\left(anX_n^t - b \cdot \frac{X_n^t}{n^{t-1}}\right)^u} = \frac{n^{tu+1}}{\left(an^t - b\right)^u X_n^u} \tag{5}$$

By (2), (3) and (5) we obtain

$$\sum_{k=1}^{n} \frac{x_k^m}{\left(aX_n^t - bx_k^t\right)^u} \ge \frac{1}{n} \cdot \frac{X_n^m}{n^{m-1}} \cdot \frac{n^{tu+1}}{\left(an^t - b\right)^u X_n^{tu}} = \frac{n^{-m+tu+1}}{\left(an^t - b\right)^u} X_n^{m-tu},$$

and we are done.

Theorem 2. *If* $a,b,M,x_k \in R_+^*, k = \overline{1,n}$, *and* $m,t,u \in [1,\infty)$ *such that* $aM^t > b\max_{1\le k\le n} x_k$, *then*

$$\sum_{k=1}^{n} \frac{x_k^m}{\left(aM^t - bx_k^t\right)^u} \ge \frac{n^{tu-m+1} X_n^m}{\left(an^t M^t - bX_n^t\right)^u} \tag{6}$$

Proof. We consider the functions $g : R_+^* \to R_+^*,\ g(x) = x^m,\ h : \left(0, \left(\frac{a}{b}\right)^{\frac{1}{t}} M\right) \to R_+^*$,

$$h(x) = \left(aM^t - bx^t\right)^{-u} \text{ and } f : \left(0, \left(\frac{a}{b}\right)^{\frac{1}{t}} M\right) \to R_+^*,\ f = gh.$$

We have $g'(x) > 0, g''(x) \geq 0, \forall x \in R_+^*, h'(x) > 0, h''(x) \geq 0,\ \forall x \in \left(0, \left(\frac{a}{b}\right)^{\frac{1}{t}} M\right)$.

Because $f'' = g''h + 2g'h' + gh''$, yields that $f''(x) \geq 0$, $\forall x \in \left(0, \left(\frac{a}{b}\right)^{\frac{1}{t}} M\right)$.

Therefore the function f is convex on $\left(0, \left(\frac{a}{b}\right)^{\frac{1}{t}} M\right)$, so we can apply *Jensen*'s inequality and we obtain that:

$$\sum_{k=1}^{n} f(x_k) \geq n \cdot f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) = n \cdot f\left(\frac{X_n}{n}\right) \Leftrightarrow \sum_{k=1}^{n} \frac{x_k^m}{\left(aM^t - bx_k^t\right)^u} \geq \frac{n^{tu-m+1} X_n^m}{\left(an^t M^t - bX_n^t\right)^u} =$$

$$= \frac{n^{tu} X_n^m}{n^{m-1}\left(an^t M^t - bX_n^t\right)^u} = \frac{n^{tu-m+1} X_n^m}{\left(an^t M^t - bX_n^t\right)^u},$$

and the proof is complete.

3. APPLICATIONS

A.1. If we take in theorem 1, $n = 4$ and $x_k = S_k$, $k = \overline{1,4}$, where S_k is the area of the face opposite to the vertex A_k of the tetrahedron $[A_1 A_2 A_3 A_4]$, then $X_4 = S =$ the area of the tetrahedron, and (IMO) becomes

$$\sum_{k=1}^{4} \frac{S_k^m}{\left(aS^t - bS_k^t\right)^u} \geq \frac{4^{-m+tu+1}}{\left(4^t a - b\right)^u} S^{m-tu} \tag{7}$$

A.2. If we take in theorem 1, $n = 3$ and $x_1 = a_1, x_2 = a_2, x_3 = a_3$, where $a_k, k = \overline{1,3}$, are the lengths of the sides of triangle $A_1 A_2 A_3$ with the perimeter $2p$, then by (IMO) we obtain that:

$$\sum_{k=1}^{3} \frac{a_k^m}{\left(a \cdot 2^t \cdot p^t - b \cdot a_k^t\right)^u} \geq \frac{3^{-m+tu+1}}{\left(3^t a - b\right)^u} \cdot 2^{m-tu} \cdot p^{m-tu},$$

where if we putting $a = b = t = u = 1$, yields that

$$\sum_{k=1}^{3} \frac{a_k^m}{2p - a_k} \geq \left(\frac{2}{3}\right)^{m-2} p^{m-1},$$

i.e. (1).

A.3. If in theorem 2, we consider a convex polygon with the perimeter $2p$ and the sides $[A_k A_{k+1}]$ with the lenghts x_k, $k=\overline{1,n}$, and $M = p$, we obtain:

$$\sum_{k=1}^{n} \frac{x_k^m}{\left(ap^t - bx_k^t\right)^u} \geq \frac{n^{tu-m+1} \cdot 2^m \cdot p^m}{\left(a \cdot n^t \cdot p^t - b \cdot 2^t \cdot p^t\right)^u} = \frac{n^{tu-m+1} \cdot 2^m \cdot p^{m-tu}}{\left(a \cdot n^t - b \cdot 2^t\right)^u} \tag{8}$$

A.4. If in A.3. a convex quadrilateral, then

$$\sum_{k=1}^{4} \frac{x_k^m}{\left(ap^t - bx_k^t\right)^u} \geq \frac{4^{tu-m+1} \cdot 2^m \cdot p^m}{\left(a \cdot 4^t \cdot p^t - b \cdot 2^t \cdot p^t\right)^u} = \frac{2^{tu-m+2} \cdot p^{m-tu}}{\left(2^t \cdot a - b\right)^u},$$

where if we putting $a = b = t = u = 1$, we deduce that:

$$\sum_{k=1}^{4} \frac{x_k^m}{p - x_k} \geq 2^{-m+3} \cdot p^{m-1} \tag{9}$$

A.5. If in theorem 2, we consider the tetrahedron $[A_1 A_2 A_3 A_4]$ with the total area S and S_k, $k=\overline{1,4}$ the area of the faces opposite to the vertex A_k, then by (6) we obtain for $M = S$ that:

$$\sum_{k=1}^{4} \frac{S_k^m}{\left(aS^t - bS_k^t\right)^u} \geq \frac{4^{tu-m+1} \cdot S^m}{\left(a \cdot 4^t \cdot S^t - b \cdot S^t\right)^u} = \frac{4^{tu-m+1} \cdot S^{m-tu}}{\left(4^t \cdot a - b\right)^u} \tag{10}$$

where if we putting $a = b = t = u = 1$, yields:

$$\sum_{k=1}^{4} \frac{S_k^m}{S^t - S_k^t} \geq \frac{4^{-m+2} \cdot S^{m-1}}{3} \tag{11}$$

A dual application of IMO. *If a, b, c, are the lengths of the sides of a triangle ABC, with the perimeter $2p$, then*:

$$a\left(\frac{1}{b^m} + \frac{1}{c^m}\right) + b\left(\frac{1}{c^m} + \frac{1}{a^m}\right) + c\left(\frac{1}{a^m} + \frac{1}{b^m}\right) \geq \frac{3^m}{2^{m-2} p^{m-1}}, \forall m \in [1, \infty) \tag{$1'$}$$

We observe that ($1'$) is equivalent with:

$$a\left(\frac{1}{b^m} + \frac{1}{c^m}\right) + b\left(\frac{1}{c^m} + \frac{1}{a^m}\right) + c\left(\frac{1}{a^m} + \frac{1}{b^m}\right) \geq \frac{2 \cdot 3^m}{(a+b+c)^{m-1}} \tag{$1''$}$$

Proof 1. We have:

$$W = a\left(\frac{1}{b^m} + \frac{1}{c^m}\right) + b\left(\frac{1}{c^m} + \frac{1}{a^m}\right) + c\left(\frac{1}{a^m} + \frac{1}{b^m}\right) = \frac{a}{b^m} + \frac{b}{a^m} + \frac{b}{c^m} + \frac{c}{b^m} + \frac{c}{a^m} + \frac{a}{c^m} =$$

$$=\frac{a^{m+1}+b^{m+1}}{(ab)^m}+\frac{b^{m+1}+c^{m+1}}{(bc)^m}+\frac{c^{m+1}+a^{m+1}}{(ca)^m},$$

where we apply the inequality of *J. Radon*, i.e.

$$\frac{x_1^{m+1}}{y_1^m}+\frac{x_2^{m+1}}{y_2^m}+\frac{x_3^{m+1}}{y_3^m}\ge\frac{(x_1+x_2+x_3)^{m+1}}{(y_1+y_2+y_3)^m},\ \forall x_k,y_k\in R_+^*,\ k=\overline{1,3} \qquad \text{(R)}$$

and we obtain that:

$$W\ge\frac{2(a+b+c)^{m+1}}{(ab+bc+ca)^m},\ \forall a,b,c\in R_+^* \qquad (12)$$

Since, $(a+b+c)^2\ge 3(ab+bc+ca),\ \forall a,b,c\in R_+^*$, then by (12) we deduce that:

$$W\ge\frac{2(a+b+c)^{m+1}}{(a+b+c)^{2m}}\cdot 3^m=\frac{2\cdot 3^m}{(a+b+c)^{m-1}},$$

and we are done.

Proof 2. We have:

$$W=\frac{b+c}{a^m}+\frac{c+a}{b^m}+\frac{a+b}{c^m}=\frac{(b+c)^{m+1}}{\left(a(b+c)\right)^m}+\frac{(c+a)^{m+1}}{\left(b(c+a)\right)^m}+\frac{(a+b)^{m+1}}{\left(c(a+b)\right)^m},$$

and by (R) yields that:

$$W\ge\frac{2^{m+1}(a+b+c)^{m+1}}{2^m(ab+bc+ca)^m}=\frac{2(a+b+c)^m}{(ab+bc+ca)^m} \qquad (13)$$

Since:

$$ab+bc+ca\le\frac{(a+b+c)^2}{3}\Leftrightarrow\frac{1}{ab+bc+ca}\ge\frac{3}{(a+b+c)^2},\ \forall a,b,c\in R_+^*, \text{ then}$$

by (13) we obtain the desired result.

Proof 3. We have:

$$W=\frac{b+c}{a^m}+\frac{c+a}{b^m}+\frac{a+b}{c^m},$$

and by AM-GM inequality, we deduce that:

$$W\ge 3\cdot\sqrt[3]{\frac{(b+c)(c+a)(a+b)}{a^mb^mc^m}}\ge 3\cdot\sqrt[3]{\frac{2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}}{(abc)^m}}=3\cdot\sqrt[3]{\frac{8}{(abc)^{m-1}}}=\frac{6}{\sqrt[3]{(abc)^{m-1}}} \qquad (14)$$

Also by AM-GM inequality we have:

$$a+b+c \geq 3\cdot\sqrt[3]{abc}\,, \forall a,b,c \in R_+^*, \tag{15}$$

so

$$\frac{1}{\sqrt[3]{abc}} \geq \frac{3}{a+b+c},\ \forall a,b,c \in R_+^*,$$

and then:

$$\frac{1}{\sqrt[3]{(abc)^{m-1}}} \geq \frac{3^{m-1}}{(a+b+c)^{m-1}},\ \forall m \in [1,\infty) \tag{16}$$

By (14) and (16) hence:

$$W \geq \frac{6\cdot 3^{m-1}}{(a+b+c)^{m-1}} = \frac{2\cdot 3^m}{(a+b+c)^{m-1}},$$

which finished the proof.

Proof 4. WLOG, we can assume that $a \leq b \leq c$, and then $\frac{1}{a^m} \geq \frac{1}{b^m} \geq \frac{1}{c^m}$ and $b+c \geq c+a \geq a+b$.

Applying *Chebyshev*'s inequality we obtain that:

$$W = \frac{b+c}{a^m} + \frac{c+a}{b^m} + \frac{a+b}{c^m} \geq \frac{1}{3}\cdot(b+c+c+a+a+b)\cdot\left(\frac{1}{a^m}+\frac{1}{b^m}+\frac{1}{c^m}\right) =$$
$$= \frac{2}{3}\cdot(a+b+c)\cdot\left(\frac{1}{a^m}+\frac{1}{b^m}+\frac{1}{c^m}\right),$$

and by (R) hence:

$$W \geq \frac{2}{3}\cdot(a+b+c)\cdot\frac{(1+1+1)^{m+1}}{(a+b+c)^m} = \frac{2\cdot 3^m}{(a+b+c)^{m-1}},$$

and the proof is complete.