ORIGINAL PAPER

ONE INEQUALITY AND SOME APPLICATIONS

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Abstract. *This note presents a general inequality and some aplications for this inequality.*

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1. INTRODUCTION

In the following, we deal with some methods how you can solve a class of problems which appeared in several problem solving math journals, for e.g. Problem 11634 from the American Mathematical Monthly, March 2012, i.e. If

$a,m\in R_+$, $b,c,d,x_k\in R_+^*$, $\forall k=\overline{1,n}$, $X_n=\sum_{k=1}^{n}x_k$, $p\in[1,\infty)$ and $cX_n^p>d\max_{1\le k\le n}x_k^p$,

then:

$$\sum_{k=1}^{n}\frac{aX_n+bx_k}{(cX_n^p-dx_k^p)^m}\geq\frac{(an+b)n^{mp}}{(cn^p-d)^m}X_n^{1-mp}.$$

Also see the recently, Problem 20 from MathProblems, Volume 3, Issue 1 (2013), p.119 (a solutions is presented below in the applications 6, and other yields by Theorem from Main Results).

2. MAIN RESULTS

Theorem. If $n\in N^*-\{1\}$, $a\in R_+$, $b,c,d,x_k\in R_+^*$, $X_n=\sum_{k=1}^{n}x_k$, $cX_n>d\max_{1\le k\le n}x_k$ and $m\in[1,\infty)$, $p\in R_+^*$, then:

$$\sum_{k=1}^{n}\frac{(aX_n+bx_k)^m}{(cX_n-dx_k)^p}\geq\frac{(an+b)^m}{(cn-d)^p}n^{p-m+1}X_n^{m-p} \qquad (1)$$

Proof. First we note that by $cX_n>d\max_{1\le k\le n}x_k \Rightarrow cX_n>dx_k,\ \forall k=\overline{1,n}\Rightarrow$

$$\Rightarrow\sum_{k=1}^{n}cX_n>d\sum_{k=1}^{n}x_k\Leftrightarrow cnX_n>dX_n\Leftrightarrow cn>d\ .$$

If $x_k=X_ny_k$, $k=\overline{1,n}$, then $Y_n=\sum_{k=1}^{n}y_k=1$, so

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$$U_n = \sum_{k=1}^{n} \frac{(aX_n + bx_k)^m}{(cX_n - dx_k)^p} = \frac{X_n^m}{X_n^p} \sum_{k=1}^{n} \frac{(a + by_k)^m}{(c - dy_k)^p} = X_n^{m-p} \sum_{k=1}^{n} \frac{(a + by_k)^m}{(c - dy_k)^p} \tag{2}$$

We consider the functions

$$f, g, h : \left(0, \frac{c}{d}\right) \to R_+^*,\ f(x) = (a + bx)^m,\ g(x) = (c - dx)^{-p},\ h(x) = f(x)g(x).$$

Since $h'' = f''g + 2f'g' + fg''$, and $f(x), g(x), f'(x), g'(x), f''(x), g''(x) \geq 0$, $\forall x \in \left(0, \frac{c}{d}\right)$, we have $h''(x) > 0$, $\forall x \in \left(0, \frac{c}{d}\right)$, therefore h is convex on $\left(0, \frac{c}{d}\right)$, and then we can apply the inequality of *Jensen*.

So, $h(y_1) + h(y_2) + ... + h(y_n) \geq nh\left(\frac{y_1 + y_2 + ... + y_n}{n}\right) = nh\left(\frac{1}{n}\right)$, i.e.

$$\sum_{k=1}^{n} \frac{(a + by_k)^m}{(c - dy_k)^p} \geq n \cdot \frac{\left(a + \frac{b}{n}\right)^m}{\left(c - \frac{d}{n}\right)^p} = n^{p-m+1} \cdot \frac{(an + b)^m}{(cn - d)^p} \tag{3}$$

By (2) and (3) we deduce (1).

3. APPLICATIONS

A 1. If $m = p + 1$, then (1) becomes

$$\sum_{k=1}^{n} \frac{(aX_n + bx_k)^{p+1}}{(cX_n - dx_k)^p} \geq \frac{(an + b)^{p+1}}{(cn - d)^p} X_n \tag{4}$$

Other Proof. By the inequality of *Radon* we have

$$V_n = \sum_{k=1}^{n} \frac{(aX_n + bx_k)^{p+1}}{(cX_n - dx_k)^p} \geq \frac{\left(\sum_{k=1}^{n} (aX_n + bx_k)\right)^{p+1}}{\left(\sum_{k=1}^{n} (cX_n - dx_k)\right)^p} = \frac{(anX_n + bX_n)^{p+1}}{(cnX_n - dX_n)^p} = \frac{(an + b)^{p+1}}{(cn - d)^p} X_n. \qquad \square$$

A 2. If $m = 1$, then (1) becomes

$$\sum_{k=1}^{n} \frac{aX_n + bx_k}{(cX_n - dx_k)^p} \geq \frac{an + b}{(cn - d)^p} n^p X_n^{1-p} \tag{5}$$

Other Proof. By the inequality of *Radon* we have that:

$$W_n = \sum_{k=1}^{n} \frac{aX_n + bx_k}{(cX_n - dx_k)^p} = \sum_{k=1}^{n} \frac{(aX_n + bx_k)^{p+1}}{((aX_n + dx_k)(cX_n - dx_k))^p} = \sum_{k=1}^{n} \frac{(aX_n + bx_k)^{p+1}}{(acX_n^2 + (bc - ad)X_n x_k - bdx_k^2)^p} \geq$$

$$\geq \frac{\left(\sum_{k=1}^{n} (aX_n + bx_k)\right)^{p+1}}{\left(\sum_{k=1}^{n} (acX_n^2 + (bc - ad)X_n x_k - bdx_k^2)\right)^p} = \frac{(anX_n + bX_n)^{p+1}}{\left(acnX_n^2 + (bc - ad)X_n^2 - bd\sum_{k=1}^{n} x_k^2\right)^p},$$

Where we taking into account by AM-QM inequality, i.e. $\sum_{k=1}^{n} x_k^2 \geq \frac{X_n^2}{n}$, and we obtain

$$W_n \geq \frac{(an+b)^{p+1} X_n^{p+1}}{\left(acn+bc-ad-\frac{bd}{n}\right)^p X_n^{2p}} = \frac{(an+b)^{p+1}}{(acn^n+(bc-ad)n-bd)^p} X_n^{1-p} =$$

$$= \frac{(an+b)^{p+1}}{(an+b)^p (cn-d)^p} X_n^{1-p} = \frac{an+b}{(cn-d)^p} X_n^{1-p}. \qquad \square$$

A 3. If $p=1$, then (4) becomes

$$\sum_{k=1}^{n} \frac{(aX_n+bx_k)^2}{cX_n-dx_k} \geq \frac{(an+b)^2}{cn-d} X_n \tag{6}$$

Other Proof. By *Bergström*'s inequality we have

$$T_n = \sum_{k=1}^{n} \frac{(aX_n+bx_k)^2}{cX_n-dx_k} \geq \frac{\left(\sum_{k=1}^{n}(aX_n+bx_k)\right)^2}{\sum_{k=1}^{n}(cX_n-dx_k)} = \frac{(anX_n+bX_n)^2}{cnX_n-dX_n} = \frac{(an+b)^2}{cn-d} X_n. \qquad \square$$

A 4. If $a=0, b=1$, then (6) becomes

$$\sum_{k=1}^{n} \frac{x_k^2}{cX_n-dx_k} \geq \frac{X_n}{cn-d} \tag{7}$$

Other Proof. We have:

$$T_n = \sum_{k=1}^{n} \frac{x_k^2}{cX_n-dx_k} \Leftrightarrow d^2T_n = \sum_{k=1}^{n} \frac{d^2x_k^2}{cX_n-dx_k} = \sum_{k=1}^{n} \frac{d^2x_k^2-c^2X_n^2+c^2X_n^2}{cX_n-dx_k} =$$

$$= \sum_{k=1}^{n} -(cX_n+dx_k)+c^2X_n^2 \cdot \sum_{k=1}^{n} \frac{1}{cX_n-dx_k} = -cnX_n-dX_n+\frac{c^2X_n}{cn-d}(cn-d)X_n \sum_{k=1}^{n} \frac{1}{cX_n-dx_k} =$$

$$= -cnX_n-dX_n+\frac{c^2X_n}{cn-d}\left(\sum_{k=1}^{n}(cX_n-dx_k)\right)\sum_{k=1}^{n} \frac{1}{cX_n-dx_k},$$

and by AM-HM inequality we obtain

$$d^2T_n \geq -(cn+d)X_n+\frac{c^2n^2}{cn-d}X_n = \frac{d^2-c^2n^2+c^2n^2}{cn-d}X_n = \frac{d^2X_n}{cn-d} \Leftrightarrow T_n \geq \frac{X_n}{cn-d}. \qquad \square$$

A 5. If $m=p=1, a=0, b=1$, then (1) becomes

$$\sum_{k=1}^{n} \frac{x_k}{cX_n-dx_k} \geq \frac{n}{cn-d} \tag{8}$$

Other Proof. The inequality (8) yields by *Bergström*'s inequality in two ways:
First Proof

$$S_n = \sum_{k=1}^{n} \frac{x_k}{cX_n-dx_k} = \sum_{k=1}^{n} \frac{x_k^2}{cX_nx_k-dx_k^2} \geq \frac{\left(\sum_{k=1}^{n} x_k^2\right)}{cX_n^2-d\sum_{k=1}^{n} x_k^2} \overset{AM-QM}{\geq} \frac{X_n^2}{cX_n^2-\frac{d}{n}X_n^2} = \frac{n}{cn-d}. \qquad \square$$

Second Proof

$$dS_n = \sum_{k=1}^{n} \frac{dx_k}{cX_n - dx_k} = \sum_{k=1}^{n} \frac{dx_k - cX_n + cX_n}{cX_n - dx_k} = -n + cX_n \cdot \sum_{k=1}^{n} \frac{1}{cX_n - dx_k},$$

$$dS_n \geq -n + cX_n \cdot \frac{n^2}{\sum_{k=1}^{n}(cX_n - dx_k)} = -n + \frac{cn^2 X_n}{cnX_n - dX_n} = -n + \frac{cn^2}{cn-d} =$$

$$= \frac{dn}{cn-d} \Leftrightarrow S_n \geq \frac{n}{cn-d}. \qquad \square$$

Remark 1 . By (8) for $c = d = 1$, we obtain the inequality of *Nesbitt* for n variables, i.e.

$$\sum_{k=1}^{n} \frac{x_k}{X_n - x_k} \geq \frac{n}{n-1} \tag{9}$$

Finally we prove that:

A 6. If $n \in N^* - \{1\}$, $a \in R_+$, $b, c, d, x_k \in R_+^*$, $X_n = \sum_{k=1}^{n} x_k$, $cX_n > d \max_{1\leq k\leq n} x_k$, then:

$$\sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n - dx_k} \geq \frac{n(an+b)}{cn-d} \tag{10}$$

Proof. We consider the trinomial

$$T_n = \left(\sum_{k=}^{n} \sqrt{\frac{aX_n + bx_k}{cX_n - dx_k}} \cdot Y - \sqrt{(aX_n + bx_k)(cX_n - dx_k)} \right)^2 =$$

$$= \sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n - dx_k} \cdot Y^2 - 2\left(\sum_{k=1}^{n} (aX_n + bx_k) \right) Y + \sum_{k=1}^{n} (aX_n + bx_k)(cX_n - dx_k) =$$

$$= \sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n - dx_k} \cdot Y^2 - 2(an+b)X_n Y + \sum_{k=1}^{n} \left(acX_n^2 + (bc-ad)X_n x_k - bdx_k^2 \right) =$$

$$= \sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n - dx_k} \cdot Y^2 - 2(an+b)X_n Y + acnX_n^2 + (bc-ad)X_n^2 - bd\sum_{k=1}^{n} x_k^2 .$$

We note that $T_n(y) \geq 0, \forall y \in R$, so yields that

$$\Delta' = (an+b)^2 X_n^2 - \left(\sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n - dx_k} \right) \left((acn + bc - ad)X_n^2 - bd\sum_{k=1}^{n} x_k^2 \right) \leq 0 \Leftrightarrow$$

$$\Leftrightarrow \sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n - dx_k} \geq \frac{(an+b)^2 X_n^2}{\left((acn + bc - ad)X_n^2 - bd\sum_{k=1}^{n} x_k^2 \right)},$$

and if we taking account by $\sum_{k=1}^{n} x_k^2 \geq \frac{X_n^2}{n}$, we deduce that

$$\sum_{k=1}^{n} \frac{aX_n + bx_k}{cX_n - dx_k} \geq \frac{(an+b)^2 X_n^2}{\left(acn + bc - ad - \frac{bd}{n} \right) X_n^2} = \frac{(an+b)^2 n}{acn^2 + (bc-ad)n - bd} = \frac{(an+b)^2 n}{(an+b)(cn-d)} =$$

$$= \frac{(an+b)n}{cn-d}. \qquad \square$$

Remark 2. The inequality (10) yields by (1) taking $m = p = 1$.