ORIGINAL PAPER

LIE-ALGEBRA AND MODIFIED JACOBI POLYNOMIALS

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Abstract: *In the present paper, L. Weisner's group-theoretic method has been utilized in obtaining generating functions of* $P_n^{(\alpha+n,\,\beta+n)}(x)$*, a modification of the Jacobi polynomials -* $P_n^{(\alpha,\beta)}(x)$*. In section 1, with the suitable simultaneous interpretations of the index* (n) *and the parameters* (α,β) *of the modified Jacobi polynomials, a set of linear partial differential operators, their commutators and the extended form of the groups corresponding to the operators are introduced and on showing that they form a four diamentional Lie algebra, we obtain in section 3, a novel generating relation for the polynomial under consideration which in turn yields a number of generating relations most of which are seem to be new.*

Keywords: *Jacobi polynomials, generating functions.*

AMS-1991 Subject Classifications*: 33A75*

1. INTRODUCTION

Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$ defined by [1],

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\; {}_2F_1\left[\begin{matrix} -n, & 1+\alpha+\beta+n\,; \\ & 1+\alpha\,; \end{matrix}\; \frac{1-x}{2}\right] \tag{1.1}$$

satisfies the following differential equation :

$$(1-x^2)\frac{d^2v}{dx^2} + [\beta-\alpha-(2+\alpha+\beta)x]\frac{dv}{dx} + n(1+\alpha+\beta+n)v = 0. \tag{1.2}$$

In the present paper, we consider $P_n^{(\alpha+n,\,\beta+n)}(x)$, a modification of $P_n^{(\alpha,\beta)}(x)$ satisfying the ordinary differential equation:

$$(1-x^2)\frac{d^2v}{dx^2} + [\,\beta-\alpha-(2+\alpha+\beta+2n)x\,]\frac{dv}{dx} + n(1+\alpha+\beta+3n)v = 0. \tag{1.3}$$

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The object of the present paper is to obtain some novel generating functions of modified Jacobi polynomials, $P_n^{(\alpha+n,\beta+n)}(x)$ by using Weisner's group-theoretic method [2-4] (which is lucidly presented in the book "Obtaining Generating Functions" written by E.B. McBride [5]),with the suitable simultaneous interpretations of the index (n) and the parameters (α,β) of the polynomials under consideration. For previous works on Jacobi /modified Jacobi polynomials, on may refer to the works [6-18]. The main results of the investigation are given in section 3.

2. GROUP-THEORETIC DISCUSSION:

Replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, α by $y\frac{\partial}{\partial y}$, β by $z\frac{\partial}{\partial z}$, n by $t\frac{\partial}{\partial t}$ and v by $u(x,y,z,t)$ in (1.3) , we get the following partial differential equation :

$$\begin{aligned}&(1-x^2)\frac{\partial^2 u}{\partial x^2}+(1-x)z\frac{\partial^2 u}{\partial z\partial x}-(1+x)y\frac{\partial^2 u}{\partial y\partial x}-2x\frac{\partial u}{\partial x}-2xt\frac{\partial^2 u}{\partial t\partial x}+t\frac{\partial u}{\partial t}\\&+ty\frac{\partial^2 u}{\partial t\partial y}+tz\frac{\partial^2 u}{\partial t\partial z}+3t^2\frac{\partial^2 u}{\partial t^2}+3t\frac{\partial u}{\partial t}=0.\end{aligned} \tag{2.1}$$

Thus $u(x,y,z,t)=P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n$ is a solution of the differential equation (2.1) since $P_n^{(\alpha,\beta)}(x)$ is a solution of (1.3).

We now define a set of infinitesimal operators, A_i $(i=1,2,3,4,5)$:

$$A_i = A_i^{(1)}\frac{\partial}{\partial x}+A_i^{(2)}\frac{\partial}{\partial y}+A_i^{(3)}\frac{\partial}{\partial z}+A_i^{(4)}\frac{\partial}{\partial t}+A_i^{(0)}$$

as follows

$$\begin{cases} A_1 = y\frac{\partial}{\partial y} \quad ; \quad A_2 = z\frac{\partial}{\partial z} \\ A_3 = t\frac{\partial}{\partial t} \quad ; \quad A_4 = y^2 z^2 t^{-1}\frac{\partial}{\partial x} \\ A_5 = (1-x^2)y^{-2}z^{-2}t\frac{\partial}{\partial x}-(1+x)y^{-1}z^{-2}t\frac{\partial}{\partial y}+(1-x)y^{-2}z^{-1}t\frac{\partial}{\partial z}-2xy^{-2}z^{-2}t^2\frac{\partial}{\partial t} \end{cases} \tag{2.2}$$

such that

$$\begin{cases} A_1[P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n]=\alpha\,P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n \\ A_2[P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n]=\beta P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n \\ A_3[P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n]=nP_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n \\ A_4[P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n]=\frac{1}{2}(1+\alpha+\beta+3n)P_{n-1}^{(\alpha+n+1,\beta+n+1)}(x)\,y^{\alpha+2}z^{\beta+2}t^{n-1} \\ A_5[P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n]=-2(n+1)P_{n+1}^{(\alpha+n-1,\beta+n-1)}(x)\,y^{\alpha-2}z^{\beta-2}t^{n+1}. \end{cases} \tag{2.3}$$

We now proceed to find the commutator relations, using the notation:

$$[A,B]u = [AB-BA]u$$

we have,

$$\begin{cases}[A_i, A_j] = (-1)^j 2A_j \quad i=1,2 \quad j=4,5 \quad ; \quad [A_3, A_i] = (-1)^{i+1} A_i \quad i=4,5 \\ [A_i, A_j] = 0 \qquad i=1 \quad j=2,3 \quad ; \quad [A_2, A_3] = 0 \\ [A_4, A_5] = -(A_1 + A_2 + A_3) \ . \end{cases} \tag{2.4}$$

From the above commutator relations, one can state the following theorem.

Theorem: The set of operators $\{1, A_i \ (i = 1,2,3,4,5)\}$, where 1 stands for the identity operator, generates a Lie-algebra $\mathscr{L}$.

It can be easily shown that the partial differential operator L given by,

$$L\,u = (1-x^2)\frac{\partial^2 u}{\partial x^2} + (1-x)z\frac{\partial^2 u}{\partial z \partial x} - (1+x)y\frac{\partial^2 u}{\partial y \partial x} - 2x\frac{\partial u}{\partial x} - 2xt\frac{\partial^2 u}{\partial t \partial x} + t\frac{\partial u}{\partial t}$$
$$+ ty\frac{\partial^2 u}{\partial t \partial y} + tz\frac{\partial^2 u}{\partial t \partial z} + 3t^2\frac{\partial^2 u}{\partial t^2} + 3t\frac{\partial u}{\partial t},$$

which can be related to the A_i in the following way

$$L = A_5 A_4 + A_3(1 + A_1 + A_2 + 3A_3),$$

commutes with $A_i \ (i = 1,2,3,4,5)$ i.e .

$$[A_i, L] = 0, \qquad i = 1,2,3,4,5. \tag{2.5}$$

The extended form of the groups generated by $A_i \ (i = 1,2,3,4,5)$ are given by

$$e^{a_1 A_1} f(x,y,z,t) = f(x, e^{a_1} y, z, t) \tag{2.6}$$

$$e^{a_2 A_2} f(x,y,z,t) = f(x, y, e^{a_2} z, t) \tag{2.7}$$

$$e^{a_3 A_3} f(x,y,z,t) = f(x, y, z, e^{a_3} t) \tag{2.8}$$

$$e^{a_4 A_4} f(x,y,z,t) = f(x + a_4 y^2 z^2 t^{-1}, y, z, t) \tag{2.9}$$

$$\begin{aligned} e^{a_5 A_5} f(x,y,z,t) = f\big(& x + a_5(1-x^2)y^{-2}z^{-2}t,\ y\{1-a_5(1+x)y^{-2}z^{-2}t\},\ z\{1+a_5(1-x)y^{-2}z^{-2}t\}, \\ & t\{1-a_5(1+x)y^{-2}z^{-2}t\}\{1+a_5(1-x)y^{-2}z^{-2}t\}\big), \end{aligned} \tag{2.10}$$

where a_i is arbitrary .

From above, we easily get,

$$\begin{aligned} & e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1} f(x,y,z,t) \\ & = f\left(x+a_5(1-x^2)y^{-2}z^{-2}t+a_4y^2z^2t^{-1}\{1-a_5(1+x)y^{-2}z^{-2}t\}\{1+a_5(1-x)y^{-2}z^{-2}t\},\right. \\ & e^{a_1}y\{1-a_5(1+x)y^{-2}z^{-2}t\}, e^{a_2}z\{1+a_5(1-x)y^{-2}z^{-2}t\}, \\ & \left. e^{a_3}t\{1-a_5(1+x)y^{-2}z^{-2}t\}\{1+a_5(1-x)y^{-2}z^{-2}t\}\right). \end{aligned} \tag{2.11}$$

3. GENERATING FUNCTIONS:

From (2.1), we find that $u(x,y,z,t) = P_n^{(\alpha+n,\beta+n)}(x)y^\alpha z^\beta t^n$ is a solution of the system:

$$L\,u=0 \qquad L\,u=0 \qquad L\,u=0$$

$$\left(A_1-\alpha\right)u=0; \qquad \left(A_2-\beta\right)=0; \qquad \left(A_3-n\right)u=0$$

$$L\,u=0$$

$$\left(A_1+A_2+A_3-\alpha-\beta-n\right)u=0.$$

From (2.5), we easily get

$$S\,L\left(P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n\right) = L\,S\,\left(P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n\right) = 0,$$

where

$$S = e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}.$$

Therefore the transformation $S\left(P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n\right)$ is annihilated by L .

By setting $a_1=0, a_2=0$, $a_3=0$ and writing $f(x,y,z,t) = P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n$ in (2.11), we get

$$\begin{aligned} & e^{a_5A_5}e^{a_4A_4}\left(P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n\right) \\ & = \left\{1-a_5(1+x)y^{-2}z^{-2}t\right\}^{\alpha+n}\left\{1+a_5(1-x)y^{-2}z^{-2}t\right\}^{\beta+n} y^\alpha z^\beta t^n \\ & \times P_n^{(\alpha+n,\beta+n)}\left(x+a_5(1-x^2)y^{-2}z^{-2}t + a_4y^2z^2t^{-1}\left\{1-a_5(1+x)y^{-2}z^{-2}t\right\}\left\{1+a_5(1-x)y^{-2}z^{-2}t\right\}\right). \end{aligned} \tag{3.1}$$

On the other hand

$$\begin{aligned} & e^{a_5A_5}e^{a_4A_4}\left(P_n^{(\alpha+n,\beta+n)}(x)\,y^\alpha z^\beta t^n\right) = \\ & = \sum_{k=0}^{\infty}\sum_{p=0}^{n+k} \frac{\left(-2a_5y^{-2}z^{-2}t\right)^k}{k!}\frac{\left(\frac{a_4}{2}y^2z^2t^{-1}\right)^p}{p!}\left(1+\alpha+\beta+3n\right)_p\left(n-p+1\right)_k \\ & \times P_{n-p+k}{}^{(\alpha+n+p-k,\beta+n+p-k)}(x)\,y^\alpha z^\beta t^n. \end{aligned} \tag{3.2}$$

Equating (3.1) and (3.2), we get

$$\left\{1-a_5(1+x)y^{-2}z^{-2}t\right\}^{\alpha+n}\left\{1+a_5(1-x)y^{-2}z^{-2}t\right\}^{\beta+n}$$
$$\times P_n^{(\alpha+n,\beta+n)}\left(x+a_5(1-x^2)y^{-2}z^{-2}t+a_4y^2z^2t^{-1}\left\{1-a_5(1+x)y^{-2}z^{-2}t\right\}\left\{1+a_5(1-x)y^{-2}z^{-2}t\right\}\right)$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{\left(-2a_5y^{-2}z^{-2}t\right)^k}{k!}\frac{\left(\frac{a_4}{2}y^2z^2t^{-1}\right)^p}{p!}\left(1+\alpha+\beta+3n\right)_p\left(n-p+1\right)_k \tag{3.3}$$
$$\times P_{n-p+k}^{(\alpha+n+p-k,\beta+n+p-k)}(x),$$

which does not seem to appear before. Here we would like to point it out that a good number of generating relations can be obtained from (3.3) by attributing different values to a_4 & a_5.

Before discussing the particular cases, it is of interest to mention that the operators A_4, A_5 being non commutative, the relation (3.3) will change, if we change the order of $e^{a_5A_5}e^{a_4A_4}$.

Now we consider the following cases:

Case 1: Putting $a_4=0$, $a_5=1$, $-2a_5y^{-2}z^{-2}t=w,$ then $a_5y^{-2}z^{-2}t=-\frac{w}{2}$, we get

$$\sum_{k=0}^{\infty}\frac{w^k}{k!}(n+1)_k\,P_{n+k}^{(\alpha+n-k,\beta+n-k)}(x)=$$
$$=\left\{1+\frac{w}{2}(1+x)\right\}^{\alpha+n}\left\{1-\frac{w}{2}(1-x)\right\}^{\beta+n}P_n^{(\alpha+n,\beta+n)}\left(x-\frac{w}{2}\left(1-x^2\right)\right) \tag{3.4}$$

Special Case: Putting $n=0$, we get

$$\left\{1+\frac{w}{2}(1+x)\right\}^{\alpha}\left\{1-\frac{w}{2}(1-x)\right\}^{\beta}=\sum_{k=0}^{\infty}w^kP_k^{(\alpha-k,\beta-k)}(x) \tag{3.5}$$

which is found derived in a paper of W.A.Al-Salam [19] .

Case 2: Putting $a_5=0$, $a_4=1$, and $\frac{a_4y^2z^2t^{-1}}{2}=w$, we get

$$\sum_{p=0}^{n+k}\frac{w^p}{p!}\left(1+\alpha+\beta+3n\right)_p P_{n-p}^{(\alpha+n+p,\beta+n+p)}(x)=P_n^{(\alpha+n,\beta+n)}(x+2w). \tag{3.6}$$

Replacing $2w$ by t_1, we get

$$\sum_{p=0}^{n+k}\frac{\left(\frac{t_1}{2}\right)^p}{p!}\left(1+\alpha+\beta+3n\right)_p P_{n-p}^{(\alpha+n+p,\beta+n+p)}(x)=P_n^{(\alpha+n,\beta+n)}\left(x+t_1\right). \tag{3.7}$$

Case -3: Putting $a_5 = 1$, $a_4 = -\frac{1}{w_1}$ and $y^{-2}z^{-2}t = w$, then we get

$$\begin{aligned}&\{1-w(1+x)\}^{\alpha+n}\{1+w(1-x)\}^{\beta+n}\\&\times P_n^{(\alpha+n,\beta+n)}\left(x+w(1-x^2)-\frac{1}{ww_1}\{1-w(1+x)\}\{1+w(1-x)\}\right)\\&=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(-2w)^{k-p}\left(\frac{1}{w_1}\right)^p}{k!\,p!}(1+\alpha+\beta+3n)_p\,(n-p+1)_k\,P_{n-p+k}^{(\alpha+n+p-k,\beta+n+p-k)}(x).\end{aligned} \tag{3.8}$$

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REFERENCES

[1] Renville, E.D., *Special functions*, Chelsea Publishing Company Bronx, New York, 1960.
[2] Weisner, L., *Pacific Jour. Math*., **5**, 1033, 1955.
[3] Weisner, L., *Canad. Jour. Math*., **11**, 141, 1959.
[4] Weisner, L., *Canad. Jour. Math*., **11**, 148, 1959.
[5] McBride, E.B., *Obtaining generating functions*, Springer Verlag, Berlin, 1971.
[6] Chakraborty, A.B., *Jour. Indian Inst. Sci*, **64B**, 97, 1983.
[7] Ghosh, B., *Bull.Cal. Math. Soc*., **75**(4), 227, 1983.
[8] Ghosh, B., *Pure Math. Manuscript*, **3**, 139, 1984.
[9] Ghosh, B., *Pure Math. Manuscript*, **5**, 21, 1986.
[10] Ghosh, B., *Pure Math. Manuscript*, **4**, 5, 1985.
[11] Chongdar, A.K., *Bull.Cal.Math.Soc*., **78**, 363, 1985.
[12] Chongdar, A.K., *Bull.Cal. Math. Soc*., **77**(3), 151, 1985.
[13] Chongdar, A.K., Some generating functions of Jacobi polynomials from the view point of Lie-group, communicated, 1985.
[14] Chongdar, A.K., Chatterjea, S.K. *Bull.Cal.Math.Soc*., **73**(3), 127, 1981.
[15] Sharma, R., Chongdar A.K., *Bull.Inst.Math.Acad.Sincia*, **20**(3), 230, 1992.
[16] Das,M. K., *Proceedings of the Amer.Math. Soc*., **32**(2), 565, 1972.
[17] GuhaThakurata, B.K., *Proc. Ind. Acad.Sci. (Math. Sci.)*, **95**(1), 53, 1986.
[18] Feldhim, E., *Acta Mathematica*, **75**, 117, 1943.
[19] Al-Salam,W.A., *Duke Math. Jour*., **31**, 127, 1964.