ORIGINAL PAPER

ON THE EXTENSION OF BILATERAL GENERATING FUNCTIONS OF MODIFIED JACOBI POLYNOMIALS

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Abstract. *In this note, we have obtained a novel extension of a bilateral generating function of modified Jacobi polynomials from the existence of quasi-bilateral generating function by group-theoretic method.*

Keywords: *Bilateral generating relations, Modified Jacobi Polynomials.*

AMS Subject Classification: *33A65, 33A75.*

1. INTRODUCTION

In [1], the quasi bilateral /(bilinear) generating function is defined by

$$G(x, u, w) = \sum_{n=0}^{\infty} a_n \; p_n^{(\alpha)}(x) \; q_m^{(n)}(u) \, w^n, \tag{1.1}$$

where a_n, the co-efficients are quite arbitrary and $p_n^{(\alpha)}(x)$, $q_m^{(n)}(u)$ are two particular special functions of orders n, m and of parameters α and n respectively. If, in particular, $q_m^{(n)}(u) \equiv p_m^{(n)}(u)$, then the generating relation is known as quasi bilinear.

In [2], Das and Chongdar have proved the following theorem on bilateral generating function involving $P_n^{(\alpha+n,\ \beta+n)}(x)$, a modification of Jacobi polynomials[3] by group-theoretic method introduced by Weisner[4] which is lucidly presented in the monograph by E.B. McBride[5].

Theorem 1: If there exists a unilateral generating function

$$G(x, w) = \sum_{n=0}^{\infty} a_n \, P_n^{(\alpha+n,\ \beta+n)}(x) \; w^n \tag{1.2}$$

then

$$\left\{1+\frac{w}{2}(1+x)\right\}^{\alpha} \left\{1-\frac{w}{2}(1-x)\right\}^{\beta} G\left(x-\frac{w}{2}(1-x^2),\ wv\,\{1+\frac{w}{2}(1+x)\}\,\{1-\frac{w}{2}(1-x)\} \right) = \sum_{n=0}^{\infty} w^n \, \sigma_n(x, v) \tag{1.3}$$

where

$$\sigma_n(x, v) = \sum_{p=0}^{n} a_p \binom{n}{p} P_n^{(\alpha-n+2p,\ \beta-n+2p)}(x) \; v^p \tag{1.4}$$

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The importance of the above theorem lies in the fact that whenever a unilateral generating relation of the form (1.2) is known, then the corresponding bilateral generating relation can at once be written down from (1.3). Thus a large number of bilateral generating relations can be obtained by attributing different values to a_n in (1.2). The aim at writing this note is to show that the existence of a quasi- bilinear generating function implies the existence of a more general generating function from the group theoretic view point. In the present paper, we have obtained the following extension of the theorem-1 from the existence of quasi bilateral generating relation.

Theorem 2. If there exists a generating function

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n\ P_n^{(\alpha+n,\ \beta+n)}(x)\, P_m^{(n,\ \beta)}(u)\ w^n \tag{1.5}$$

then

$$(1-w)^{-1-\beta-m}\ \{1-w(1+x)\}^{\alpha}\ \{1+w(1-x)\}^{\beta} \times G\left(x+w(1-x^2),\ \frac{u+w}{1-w},\ \frac{wv}{1-w}\ \{1-w(1+x)\}\ \{1+w(1-x)\}\right)$$
$$=\sum_{n,p,q=0}^{\infty} a_n\ \frac{w^{p+q+n}\ (-2)^p}{p!\ q!}(n+1)_p\ P_{n+P}^{(\alpha+n-p,\ \beta+n-p)}(x)\ (1+n+\beta+m)_q P_m^{(n+q,\ \beta)}(u) v^n. \tag{1.6}$$

2. PROOF OF THE THEOREM 2

For the modified Jacobi polynomial, we consider the following operators [2, 6]

$$R_1=(1-x^2)y^{-2}z^{-2}t\frac{\partial}{\partial x}-(1+x)y^{-1}z^{-2}t\frac{\partial}{\partial y}-\ (1-x)y^{-2}z^{-1}t\frac{\partial}{\partial z}-2xy^{-2}z^{-2}t^2\ \frac{\partial}{\partial t},$$

$$R_2=(1+u)\ \zeta\ \frac{\partial}{\partial u}+\zeta^2\ \frac{\partial}{\partial \zeta}+(1+\beta+m)\zeta.$$

such that

$$R_1\left(P_n^{(\alpha+n,\ \beta+n)}(x)\ y^{\alpha}\ z^{\beta}\ t^n\right)=-2(n+1)\ P_{n+1}^{(\alpha+n-1,\ \beta+n-1)}(x)\ y^{\alpha-2}\ z^{\beta-2}\ t^{n+1} \tag{2.1}$$

$$R_2\left(P_m^{(n,\ \beta)}(u)\ \zeta^n\right)=(1+n+\beta+m)P_m^{(n+1,\ \beta)}(u)\ \ \zeta^{n+1} \tag{2.2}$$

and

$$e^{wR_1} f(x,y,z,t)=f\left(x+w(1-x^2)y^2z^2t,\ \ y\{1-w(1+x)y^{-2}z^{-2}t\},\ z\{1+w(1-x)y^{-2}z^{-2}t\},\ t\{1-w(1+x)y^{-2}z^{-2}t\}\ \{1+w(1-x)y^{-2}z^{-2}t\}\right), \tag{2.3}$$

$$e^{wR_2} f(u,\zeta)=(1-w\zeta)^{-1-\beta-m}\ \ f\left(\frac{u+w\zeta}{1-w\zeta},\frac{\zeta}{1-w\zeta}\right) \tag{2.4}$$

We now consider the following formula

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n \; P_n^{(\alpha+n,\ \beta+n)}(x)\, P_m^{(n,\ \beta)}(u)\, w^n. \tag{2.5}$$

Replacing w by $wvt\,\zeta$ in (2.5) and then multiplying both sides by $y^{\alpha}\ z^{\beta}$, we get

$$y^{\alpha}\ z^{\beta} G\left(x,u,wvt\zeta\right)=\sum_{n=0}^{\infty} a_n \left(P_n^{(\alpha+n,\ \beta+n)}(x)\, y^{\alpha}\ z^{\beta}\ t^n\right)\left(P_m^{(n,\ \beta)}(u)\,\zeta^n\right)(wv)^n. \tag{2.6}$$

Now operating $e^{wR_1}e^{wR_2}$ on both sides of (2.6), we get

$$e^{wR_1}e^{wR_2}\left[y^{\alpha}\ z^{\beta}G(x,u,wvt\zeta)\right]=e^{wR_1}e^{wR_2}\left[\sum_{n=0}^{\infty} a_n \left(P_n^{(\alpha+n,\ \beta+n)}(x)\, y^{\alpha}\ z^{\beta}\ t^n\right)\left(P_m^{(n,\ \beta)}(u)\,\zeta^n\right)(wv)^n\right]. \tag{2.7}$$

The left number of (2.7), with the help of (2.3), (2.4), becomes

$$(1-w\zeta)^{-1-\beta-m}\left\{1-w(1+x)y^{-2}z^{-2}t\right\}^{\alpha}\left\{1+w(1-x)y^{-2}z^{-2}t\right\}^{\beta} y^{\alpha}\ z^{\beta} \tag{2.8}$$
$$\times G\left(x+w(1-x^2)y^2z^2t,\ \frac{u+w\zeta}{1-w\zeta},\ \frac{wvt\zeta}{1-w\zeta}\{1-w(1+x)y^{-2}z^{-2}t\}\{1+w(1-x)y^{-2}z^{-2}t\}\right)$$

The right number of (2.7), with the help of (2.1), (2.2), becomes

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{p+q+n}\ (-2)^p}{p!\,q!}(n+1)_p\ P_{n+P}^{(\alpha+n-p,\ \beta+n-p)}(x)\, y^{\alpha-2p}\ z^{\beta-2p}\ t^{n+p}\ (1+n+\beta+m)_q P_m^{(n+q,\ \beta)}(u)\,\zeta^{n+q}v^n. \tag{2.9}$$

Equating (2.8) and (2.9) and then putting $y=z=t=\zeta=1,$ we get

$$(1-w)^{-1-\beta-m}\left\{1-w(1+x)\right\}^{\alpha}\left\{1+w(1-x)\right\}^{\beta}$$
$$\times G\left(x+w(1-x^2),\ \frac{u+w}{1-w},\ \frac{wv}{1-w}\{1-w(1+x)\}\{1+w(1-x)\}\right)$$
$$=\sum_{n,\,p,\,q=0}^{\infty} a_n \frac{w^{p+q+n}\ (-2)^p}{p!\,q!}(n+1)_p\ P_{n+P}^{(\alpha+n-p,\ \beta+n-p)}(x)\ (1+n+\beta+m)_q P_m^{(n+q,\ \beta)}(u)\, v^n.$$

This completes the proof of the theorem-2.

Corollary: Putting $m=0$ in the theorem-2, we get

$$(1-w)^{-1-\beta}\left\{1-w(1+x)\right\}^{\alpha}\left\{1+w(1-x)\right\}^{\beta}$$
$$\times G\left(x+w(1-x^2),\ \frac{wv}{1-w}\{1-w(1+x)\}\{1+w(1-x)\}\right)$$
$$=\sum_{n,\,p,\,q=0}^{\infty} a_n \frac{w^{p+q+n}\ (-2)^p}{p!\,q!}(n+1)_p\ (1+n+\beta+m)_q\ P_{n+P}^{(\alpha+n-p,\ \beta+n-p)}(x)\, v^n.$$

$$= (1-w)^{-1-\beta} \sum_{n,\,p=0}^{\infty} a_n \frac{(-2w)^{n+p}}{p!} (n+1)_p \, P_{n+P}^{(\alpha+n-p,\ \beta+n-p)}(x) \left(\frac{-v}{2(1-w)} \right)^n .$$

Replacing $\left\{ \frac{-v}{2(1-w)} \right\}$ by v and $-2w$ by w, then

$$\left\{ 1+\frac{w}{2}(1+x) \right\}^{\alpha} \left\{ 1-\frac{w}{2}(1-x) \right\}^{\beta} G\left(x-\frac{w}{2}(1-x^2),\ wv\{1+\frac{w}{2}(1+x)\}\{1-\frac{w}{2}(1-x)\} \right)$$

$$= \sum_{n,\,p=0}^{\infty} a_n \frac{w^{n+p}}{p!} (n+1)_p \, P_{n+P}^{(\alpha+n-p,\ \beta+n-p)}(x)\, v^n .$$

$$= \sum_{n=0}^{\infty} w^n \sum_{p=0}^{n} a_{n-p} \frac{(n-p+1)_p}{p!} \, P_n^{(\alpha+n-2p,\ \beta+n-2p)}(x)\, v^{n-p} .$$

Hence

$$\left\{ 1+\frac{w}{2}(1+x) \right\}^{\alpha} \left\{ 1-\frac{w}{2}(1-x) \right\}^{\beta} G\left(x-\frac{w}{2}(1-x^2),\ wv\{1+\frac{w}{2}(1+x)\}\{1-\frac{w}{2}(1-x)\} \right)$$

$$= \sum_{n=0}^{\infty} w^n \, \sigma_n(x, v),$$

where

$$\sigma_n(x, v) = \sum_{p=0}^{n} a_n \binom{n}{p} P_n^{(\alpha-n+2p,\ \beta-n+2p)}(x)\, v^p ,$$

which is the bilateral generating function obtained by Das and Chongdar [2]. □

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