ORIGINAL PAPER

ON THE SPEED OF CONVERGENCE OF THE SEQUENCES

ANDREI VERNESCU[1]

Manuscript received:12.01.2013; Accepted paper: 22.02.2013

Published online: 01.03.2013.

Abstract. *The study of any nontrivial convergence of a sequence of real numbers conducts to the problem of finding the limit but also to the problem of the speed of this convergence. This speed of convergence is characterized by the first iterated limit (that supposes the existence of a function of natural variable that tends to zero for n tending to infinity, a function that appears as the first term of a sequences of such functions). A significant characterization of the first iterated limit may be given by a two sided estimate, from which the first iterated limit results immediately.*

Keywords: *Sequences, series, the constant of Napier, the harmonic sum, the constant of Euler, two sided estimate, iterated limits, asymptotic equivalence.*

Mathematics Subject Classification 2010: *26D15; 30B10; 33F05; 40A05; 40A60.*

1. INTRODUCTION

When we study any nontrivial convergence of a sequence of real numbers $(a_n)_n$, let be this $a_n \xrightarrow[(n\to\infty)]{} a$, we are interested not only by the obtaining of the limit, but also by obtaining of some information about the speed of convergence of the sequence to its limit. This speed can be characterized by the first iterated limit, which is considered respecting a function of natural variable $n \mapsto u_1(n)$ [i. e. of a sequence $(u_1(n)_n)$], such that

$$\lim_{n\to\infty} |a_n - a| u_1(n) = l \tag{1}$$

where l is a real number (finite). (The function $n \mapsto u_1(n)$ belongs to a sequence of functions.

A characterization of this type also can be obtained from a two sided estimate of the form

$$u_1(n)(l+\delta(n)) < |a_n - a| < u_1(n)(l+\eta(n)), \tag{2}$$

with $0 < \delta(n) < \eta(n)$, where $\eta(n) \to 0$, for $n \to \infty$ (which automatically gives that also we have $\delta(n) \to 0$, for $n \to \infty$).

A two sided estimate of the form (2) is more rich in information than a relation of (1) type, because (2) permits us to deduce the equality (1), but the converse affirmation doesn't hold.

A more concise writing, which shows the existence of a relation of (1) type, but which also contains less information than (1) is the one that uses the symbol O, of *Landau*, namely

[1] Valahia University of Targoviste, Faculty of Science and Arts, 130024 Targoviste, Romania.
E-mail: avernescu@gmail.com.

$$|a_n - a| = O(u_1(n)). \tag{3}$$

In this paper we will expose some of the two sided estimates of the (2) type, obtained in relation with several remarkable concrete sequences of real numbers.

2. THE SEQUENCE WHICH DEFINES THE CONSTANT OF NAPIER AND THE RELATED TO THIS

Concerning the sequence which defines the number e, also called the constant of *Napier* or the number of *Euler* and which has the general term $e_n = (1+1/n)^n$, its speed of convergence is described by a two sided estimate mentioned in the famous book of *G. Pólya* and *G. Szegö* [11], page 38, namely

$$\frac{e}{2n+2} < e - \left(1+\frac{1}{n}\right)^n < \frac{e}{2n+1} \tag{4}$$

In *G. Pólya* and *G. Szegö* [11], this result is included in a series of other results, also interesting, but the proof needs several preliminary results, which can be obtained by certain *Taylor-Maclaurin* expansions. In the work [18], published in 1988, but ready since 1985, I gave a shorter and simpler proof for the inequality (4), based only on the well known inequality of *Hermite-Hadamard* (also called of *Jensen-Hadamard*) for the convex functions

$$\varphi\left(\frac{a+b}{2}\right) < \frac{1}{b-a}\int_a^b \varphi(x)dx < \frac{\varphi(a)+\varphi(b)}{2},$$

applied to the function $t \mapsto \varphi(t) = 1/t$ on the interval $[x, x+1]$, where $x > 0,$ which is

$$\frac{2}{2x+1} < \ln(x+1) - \ln x < \frac{2x+1}{2x(x+1)}.$$

But we also can consider other standard-sequences related to the number e, obtained by modifying in a natural manner the exponent, replacing it by $n + p$, where p is any real (fixed) number, i. e. considering the sequence of general term $(1+1/n)^{n+p}$. In [11] we also can find a classical result of *I. Schur*, namely that if $p < 1/2,$ then the sequence is strictly increasing, and if $p \geq 1/2,$ then the sequence is strictly decreasing beginning from a certain rank $n_0(p),$ which depends on p (in the case p = 1/2, the monotony beginning "accurately" since n = 1).

Related to the sequence of general term $f_n = (1+1/n)^{n+1}$, we have established in [16] the two sided estimate

$$\frac{e}{2n+1} < \left(1+\frac{1}{n}\right)^{n+1} - e < \frac{e}{2n}. \tag{5}$$

Other lifelike and in a certain sense "canonical" sequences, but which converge to 1/e are the one of general term $(1-1/n)^n$, respectively $(1-1/n)^{n+1}$. Concerning these sequences, I established in my joint papers with *C. P. Niculescu* [8] and [9], the two sided estimations

$$\frac{1}{2ne} < \frac{1}{e} - \left(1-\frac{1}{n}\right)^n < \frac{1}{(2n-1)e} \tag{6}$$

$$\frac{1}{(2n-1)e} < \left(1-\frac{1}{n}\right)^{n-1} - \frac{1}{e} < \frac{1}{(2n-2)e}. \tag{7}$$

Also, concerning all the convergences $(1+1/n)^{n+p} \to e,$ we recently established that the only which has the speed of $1/n^2$ is the one corresponding to the value $p = 1/2,$ and this is described by the two sided estimation

$$\frac{e}{12(n+1)^2} < \left(1+\frac{1}{n}\right)^{n+1/2} - e < \frac{e}{12n^2}. \tag{8}$$

3. THE SEQUENCE WHICH DEFINES THE CONSTANT OF EULER

Using the usual notation for the harmonic sum of order *n*, namely

$$H_n = 1 + 1/2 + 1/3 + ... + 1/n,$$

the sequence which defines the constant of *Euler* (also called the constant of *Euler-Mascheroni*) is $\gamma_n = H_n - \ln n.$ The sequence $(\gamma_n)_n$ is strictly decreasing and lower bounded, then it is convergent to a limit which is, by definition, the constant of *Euler* γ (also denoted sometime by C).

The speed of convergence is described by the two sided estimate of [17] of 1983,

$$\frac{1}{2n+2} < \gamma - \gamma_n < \frac{1}{2n+1}. \tag{9}$$

The inequality was found again and published in Mathematical Gazette in 1991 by *R. M. Young* [26], but the author gave a different proof.

An adjacent sequence of $(\gamma_n)_n$ is the sequence of general term $\beta_n = H_n \ln(n+1)$; the term of adjacent sequence of a given monotonic one is another sequence which has the same limit as the first, the opposite sense of monotony and is generated by the same analytic expression $f(n,t)$, but for another value of the parameter *t*. In our case, $f(n,t) = H_n - \ln(n+t)$;

The sequence $(\gamma_n)_n$ corresponds to the value $t = 0$ and the sequence $(\beta_n)_n$ corresponds to the value $t = 1$. Concerning the speed of convergence of the sequence $(\beta_n)_n$, we established a two sided estimate similar to (9)

$$\frac{1}{2n+2} < \gamma - \beta_n < \frac{1}{2n+1}. \quad (10)$$

Let’s return now at the sequence which defines the constant of *Eule*r. To improve the speed of convergence to the limit, i. e. to the constant of *Euler*, several new sequences were defined.

So, denoting, with *D. W. De Temple*, $R_n = H_n - \ln(n+1/2)$, we have the following two sided estimation ([2])

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}. \quad (11)$$

In [23] we modified the general term of the sequence $(\gamma_n)_n$ by operating not on the argument of the logarithm, but on the last term of the harmonic sum H_n, namely $1/n$, which was replaced by $1/2n$. So, denoting

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1} + \frac{1}{2n} - \ln n,$$

The speed of convergence is also of the order of $1/n^2$, namely

$$\frac{1}{12(n+1)^2} < \gamma - x_n < \frac{1}{12n^2}. \quad (12)$$

Modifying deeply the argument of the logarithm, namely defining

$$T_n = H_n - \ln\left(1 + \frac{1}{2} + \frac{1}{24n}\right),$$

the following inequality holds

$$\frac{1}{48(n+1)^3} < \gamma - T_n < \frac{1}{48n^3}, \quad (13)$$

then, the speed of convergence is of the order of $1/n^3$.

But, for every $k \in \mathrm{N}$, we can obtain easy a convergence with the speed of the order of $1/n^k$ by using the sequence of general term

$$\Gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n^k} - \ln(n^k). \quad (14)$$

In recent papers, *Cristinel Mortici* defined several sequences which converge to the constant of *Euler-Mascheroni* and improved the speed of convergence to it.

We also mention the works of *Alina Sântămărian* regarding this matter.

4. THE SEQUENCE Ω_n

This sequence is related to the one which appears in the formula of *Wallis*. Denote by W_n the general term of the formula of *Wallis*, namely

$$W_n = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot \ldots \cdot 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot \ldots \cdot (2n-1)(2n+1)} \tag{15}$$

and

$$\Omega_n = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot 2n}. \tag{16}$$

I introduced this notation (16) in 1991, and I used it in my papers [10], [15], [16] and others. The following relation holds

$$W_n = \frac{1}{\Omega_n^2}\cdot\frac{1}{2n+1}. \tag{17}$$

Concerning Ω_n, we mention the inequality from *D. S. Mitrinović* and *P. M. Vasić* [4] the inequality of *Wallis*, namely

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \Omega_n < \frac{1}{\sqrt{\pi n}} \tag{18}$$

and the inequality of *D. K. Kazarinoff*

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \Omega_n < \frac{1}{\sqrt{\pi(n+1/4)}}. \tag{19}$$

In a paper [8] of 1985, the regretted professor *Laurenţiu Panaitopol* gave a refinement of (18), namely

$$\frac{1}{\sqrt{\pi(n+1/4+1/32n)}} < \Omega_n < \frac{1}{\sqrt{\pi(n+1/4)}} \; . \tag{20}$$

Using this inequality I established in [15] the speed of convergence of the sequence of the formula of *Wallis*, namely

$$\frac{\pi}{4(2n+1)}\left(1-\frac{1}{8n}\right) < \frac{\pi}{2} - W_n < \frac{\pi}{4(2n+1)}. \tag{21}$$

In the paper [10] in cooperation with *Lászlo Tóth*, also of 1991, we established the asymptotic expansion of the sequence of *Wallis*

$$W_n = \frac{\pi}{2}\left(1 - \frac{1}{4n} + \frac{5}{32n^2} - \frac{11}{128n^3} + \ldots\right) \tag{22}$$

In this paper [10] we also established the asymptotic expansion of the sequence Ω_n, namely

$$\Omega_n = \frac{1}{\sqrt{\pi n}}\left(1-\frac{1}{8n}+\frac{1}{128n^2}+\frac{5}{1024n^3}+...\right) \tag{23}$$

In the paper [16] I established several generalizations of (20) containing more terms under the square root and asymptotic expansions which continue the formula (20) with more terms.

5. THE SEQUENCE OF TRAIAN LALESCU

Consider the sequence of *Traian Lalescu*, of general term

$$L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \qquad (n \geq 2).$$

It is convergent to the real number 1/e; there is an extensive literature related to this, which doesn't constitute one of the aims of this paper. I would only mention that the problem of finding elementary solutions was raised by *Tiberiu Popoviciu* in 1971. The first two elementary solutions were given by the regretted professor *Alexandru Lupaş* in 1976 and by *Marcel Ţena* in 1978.

The speed of convergence of this sequence to its limit is given by the following inequality of [3]

$$\frac{1}{2en}\left(1-\alpha\frac{\ln^2 n}{n}\right) < L_n - \frac{1}{e} < \frac{1}{2en}, \tag{24}$$

where α is any real number greater than ¼.

We also mention an asymptotic expansion of L_n, given in [11].

6. THE SPEED OF CONVERGENCE OF THE GENERALIZED HARMONIC SERIES

Let s be a real number, $s > 1$. Denote

$$\zeta_n(s) = 1+\frac{1}{2^s}+\frac{1}{3^s}+...+\frac{1}{n^s}.$$

The limit of this sequence when *n* tends to infinity, which also can be written as the sum of a series, defines the celebrated function ζ of *Riemann*, which is very important in all mathematics. The speed of convergence of the sequence to its limit is described by the following two sided estimate

$$\frac{1}{(s-1)(n+1)^{s-1}} < \zeta(s) - \zeta_n(s) < \frac{1}{(s-1)n^{s-1}}, \tag{25}$$

which can be found in the well known treatise of *G. M. Fihtenholtz*. In this book, the inequality is obtained by considering the remainder of a certain improper integral. In a paper of 1997, I obtained another proof for this inequality, based only on the monotonicity of two adequate sequences and on the mean theorem of *Lagrange*.

The function ζ of *Riemann* is extended to the complex variable, $s = \sigma + it$, and the most interesting problems appear in this domain. The crucial problem is the celebrated hypothesis of *Riemann*, concerning the distribution of the nontrivial zeros of the function on the critical line Re(s) =1/2.

7. THE COMPENSED HARMONIC SEQUENCES OF EULER TYPE

If the real number s is not smaller than 1, then the precedent series is divergent. An important discovery of Euler was the fact that by subtraction from the partial sum $\zeta_n(s)$, (where $s \in (0,1)$) of the term inspired exactly by the primitive function of $x \mapsto f(x) = 1/x^s$, we obtain a convergent sequence. A remarkable fact is the one that this convergence remains valid for the "adherent" case of $s = 1$; of course, in this case, the primitive function changes its nature becoming the logarithm and giving the celebrated constant of *Euler*, also named of *Euler-Masheroni*. Denote that the case $s = 1/2$, surely known by *Euler*, was revisited by *Andrei G. Ioachimescu* in the first issue of Gazeta Matematică, of 1895. Denoting

$$a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} - 2\sqrt{n}$$

and $a = \lim_{n\to\infty} a_n$, I established in 1991 the two sided estimate

$$\frac{1}{2\sqrt{n+1}} < a_n - a < \frac{1}{2\sqrt{n}}.$$

The corresponding two sided estimate for the adjacent sequence, of general term

$$b_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} - 2\sqrt{n+1}$$

was established in [18].

8. CONCLUDING REMARKS

All the previous examples show several speeds of convergence of the sequences to their limits, involving a collection of these speeds. The two sided estimates conduct to the finding of the first iterated limit accordingly with the asymptotic expansions (respecting a certain sequence of functions of natural variable) where such expansions exist. In some cases these limits suggested the "first steps" in the asymptotic expansions.

REFERENCES

[1] De Temple, D.W., *Amer. Math. Monthly*, **100**, 468, 1993.
[2] Kazarinoff, D.K., *Edinburgh Math. Notes*, **40**, 19, 1956.
[3] Lupaş, A.,Vernescu, A., *G. M. Seria A*, **19**(4), 212, 2001.
[4] Mitrinović, D.S., Vasić, P.M., *Analytic Inequalities*, Springer Verlag, Berlin Heidelberg New York, 1970.
[5] Negoi, T., G. M. Seria A, **15**(2), 111, 1997.
[6] Niculescu, C.P., Vernescu, A., *JIPAM*, **5**(3), art. 55, 2004.
[7] Niculescu, C.P., Vernescu, A., *Gazeta Matematică*, **109**(4), 145, 2004.
[8] Panaitopol, L., *Gazeta Matematică*, **90**(9), 329, 1985.
[9] Pólya, G., Szegö, G., *Theorems and Problems in Analysis*, Springer Verlag, Berlin Heidelberg New York, 1970.
[10] Tóth, L., Vernescu, A., *G. M. Seria A*, **11**, 26, 1990.
[11] Tóth, L., Bencze, M., Octogon, **4**(1), 12, 1996.
[12] Vernescu, A., *Gazeta Matematică*, **2-3**, 61, 1982.
[13] Vernescu, A., *Gazeta Matematică*, **10-11**, 380, 1983.
[14] Vernescu, A., *Gazeta Matematică*, **5-6**, 206, 1988.
[15] Vernescu, A., *G. M. Seria A*, **12**(1), 7, 1991.
[16] Vernescu, A., *Sur l'approximation et le développement asymptotique de la suite de terme général* $\frac{(2n-1)!!}{(2n)!!}$, Lucrările Conferinţei Anuale a Societăţii de Ştiinţe Matematice din România, Bucuresti, 29 mai-1 iunie 1997, vol. 1, Editori M. Becheanu, I. D. Ion, A.Vernescu, Bucureşti, 1998, 205-213.
[17] Vernescu, A., *G. M. Seria A*, **17**(4), 273, 1999.
[18] Vernescu, A., *On the Similar Order of Convergence of two Adjacent Sequences Related to* $\zeta(1/2)$, WSEAS Trans. on Math., **5**(12), 1297, 2006.