ORIGINAL PAPER

SOME GEOMETRIC INEQUALITIES OF IONESCU-WEITZENBÖCK TYPE IN TRIANGLE*

DUMITRU M. BĂTINEŢU-GIURGIU[1], NECULAI STANCIU[2]

Manuscript received: 10.01.2013; Accepted paper: 20.02.2013;

Published online: 01.03.2013.

Abstract. *Some Ionescu-Weitzenböck' s type inequalities for general triangles are presented. The main tool in the proofs is Radon's inequality and of course Ionescu-Weitzenböck s inequality.*

Keywords: *Ionescu-Weitzenböck's inequality, Radon's inequality.*

1. INTRODUCTION

The authors of this article demonstrated that the *Weitzenböck*'s inequality must be named the Ionescu-Weitzenböck's inequality.

Our proof is based on [1, Problem 273]:

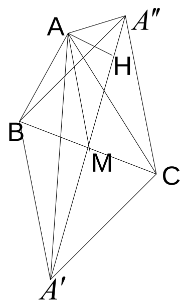
Prove that there is no triangle for which the inequality:

$$4S\sqrt{3} > a^2 + b^2 + c^2$$

be satisfied.

The solution of the problem 273, apperead in [2], as follows:

Solution by D-l N. G. Muzicescu:



Let ABC be a triangle, and we construct around the side BC the equailateral triangles BCA' and BCA''; so $M = A'A'' \cap BC$ will be the middle which of these two lines.

Let $AA' = d$ and $AA'' = d'$.

**** Paper in memoriam of Ion Ionescu (1870-1946).***

[1] *Matei Basarab* National College, 061098 Bucharest, Romania. E-mail: dmb_g@yahoo.com.

[2] *George Emil Palade* School, 120265 Buzău, Romania. E-mail: stanciuneculai@yahoo.com.

In triangle $AA'A''$, applying well known theorems we deduce that:

(1) $d^2 + d'^2 = 2 \cdot AM^2 + 2 \cdot A''M^2$

(2) $d^2 - d'^2 = 4 \cdot A''M^2 \cdot MH$;

and from triangle ABC, yields:

(3) $b^2 + c^2 = 2 \cdot AM^2 + \frac{a^2}{2}$.

On the oher hand because $A''M$ is the height of the equilateral triangle with the lengh of the side a we have:

(4) $A''M = \frac{1}{2}a\sqrt{3}$.

By the rtfelations (1), (2), (3) and (4) we deduce:

(5) $d^2 + d'^2 = a^2 + b^2 + c^2$,

(6) $d^2 - d'^2 = 2a \cdot MH\sqrt{3}$,

but: $2a \cdot MH$ reprezint 4 times the area of the triangle ABC, because MH is evidently is equal with the height of this triangle; making the substitution by (6) and (5), we have: $2d'^2 = a^2 + b^2 + c^2 - 4S\sqrt{3}$ which yields:

$$a^2 + b^2 + c^2 \geq 4S\sqrt{3},$$

So the given inequality is impossible.

Solution by: I. Moscuna and ***I. Penescu:***

We have:

$$S = \frac{1}{2}bc\sin A,\ a^2 = b^2 + c^2 - 2bc\cos A,\ \sqrt{3} = \cot g 30^0 = \frac{\cos 30^0}{\sin 30^0}.$$

Replacing , simplifying with 2, and letting in the member II only $b^2 + c^2$ we have: $bc\left(\sin A \cdot \frac{\cos 30^0}{\sin 30^0} + \cos A\right) > b^2 + c^2$, which yields:

(1) $2bc\sin(A + 30^0) > b^2 + c^2$.

On the other hand we have: $(b - c)^2 \geq 0$, so:

(2) $2bc \leq b^2 + c^2$

The inequality (1) is satisfied for $A < 150^0$, because otherwise the member I would be negative and could not be greater like the member II which is positive. Assuming this condition satisfied we can divide (1) by (2) . Therefore:

(3) $\sin(A + 30^0) > 1$.

This inequalty is impossible, the sine can not be greater than unity. So the inequality (1) and therefore the given inequality absurd for any triangle. If (3) and (2) becomes equalities, then (1) becomes equality. So must have $A + 30^0 = 90^0$, i.e. $A = 60^0$ and $b = c$, i.e. the triangle must be equilateral. Hence the given inequality becomes equality for any equilateral triangle.

Solution by Maria Rugesu and by ***Th. M. Vladimirescu***, ***G. G. Urechilă*** and ***I. Sichitiu*** and ***Corneliu P. Ionescu***.

The given inequality becomes successively:

$$4\sqrt{p(p-a)(p-b)(p-c)} \cdot \sqrt{3} > a^2 + b^2 + c^2,$$

$$16 \cdot p(p-a)(p-b)(p-c) \cdot 3 > (a^2 + b^2 + c^2)^2,$$

$$3\cdot 2p(2p-2a)(2p-2b)(2p-2c) > (a^2+b^2+c^2)^2,$$
$$3\cdot(a+b+c)(b+c-a)(a+c-b)(a+b-c) > (a^2+b^2+c^2)^2,$$
$$3\cdot[(b+c)^2-a^2][a^2-(b-c)^2] > (a^2+b^2+c^2)^2,$$
$$3\cdot[2bc+c^2+b^2-a^2][2bc+a^2-b^2-c^2] > (a^2+b^2+c^2)^2,$$
$$3\cdot[4b^2c^2-(a^2-b^2-c^2)^2] > (a^2+b^2+c^2)^2,$$
$$3\cdot[2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4] > a^4+b^4+c^4+2a^2b^2+2a^2c^2+2b^2c^2,$$
$$4a^2b^2+4a^2c^2+4b^2c^2-4a^4-4b^4-4c^4 > 0,$$
$$2[2a^4+2b^4+2c^4-2a^2b^2-2a^2c^2-2b^2c^2] < 0,$$
$$2[(a^2-b^2)^2+(a^2-c^2)^2+(b^2-c^2)^2] < 0.$$

But the last inequality is imposible, becase all the terms from the member I ar positive.

Hence, the given inequality is imposible in all triangles.

The last inequality becomes equality if and only if $a=b=c$, i.e. when the triangle is equailateral.

Also solved the above problem in different ways and by *A. Iliovici, I. Nicolaescu, E.G. Niţescu, V. V. Cambureanu* and *C. Vintilă*.

In [3] is proof that:

In any triangle ABC, with usual notations holds the inequality:

$$a^2+b^2+c^2 \geq 4\sqrt{3}S$$

We observe that the inequality of *Ion Ionescu* is the same with the inequality of *Weitzenböck*, and therefore from this moment the inequality of *Weitzenböck* must be named the inequality *Ionescu-Weitzenböck*. The inequality *Ionescu – Weitzenböck*, was given to solve at third *IMO, Veszprém, Ungaria, July 8th-15th 1961.*

A number of 11 proofs of the *Ionescu-Weitzenböck*'s inequality was presented in [4].

We discovered the above while we working on an article about *Weitzenböck*'s inequality. We also gave 23 demonstrations and 10 generalizations other than those published of the *Ionescu-Weitzenböck*'s inequality.

2. RESULTS

Theorem 1. If $m,x,y,z \in \mathrm{R}_+$ and $x+y+z \in \mathrm{R}_+^*$, then in any triangle ABC, with the area S and usual notations holds:

$$\frac{m_a^{2(m+1)}}{(xa^2+yb^2+zc^2)^m}+\frac{m_b^{2(m+1)}}{(xb^2+yc^2+za^2)}+\frac{m_c^{2(m+1)}}{(xc^2+ya^2+zb^2)} \geq \frac{3^{m+1}\sqrt{3}}{4^m(x+y+z)^m}S.$$

Proof: We have:

$$U=\sum_{cyc}\frac{m_a^{2(m+1)}}{(xa^2+yb^2+zc^2)^m}=\sum_{cyc}\frac{(m_a^2)^{m+1}}{(xa^2+yb^2+zc^2)^m},$$

and by *J. Radon'* s inequality we deduce that:

$$U \geq \frac{(m_a^2+m_b^2+m_c^2)^{m+1}}{\left(\sum_{cyc}(xa^2+yb^2+zc^2)\right)^m}=\frac{(m_a^2+m_b^2+m_c^2)^{m+1}}{(x+y+z)^m(a^2+b^2+c^2)^m}$$

But,

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2),$$

so:

$$U \geq \frac{3^{m+1}}{4^{m+1}(x+y+z)^m}(a^2 + b^2 + c^2)$$

By the inequality of *Ionescu-Weitzenböck* we have that:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S,$$

yields:

$$U \geq \frac{3^{m+1}\sqrt{3}}{4^m(x+y+z)^m}S,$$

and we are done.

Theorem 2. If $m, x, y, z \in \mathrm{R}_+$ and $x + y + z \in \mathrm{R}_+^*$, then in any triangle ABC, with the area S and usual notations holds:

$$\frac{a^{2(m+1)}}{(xm_a^2 + ym_b^2 + zm_c^2)^m} + \frac{b^{2(m+1)}}{(xm_b^2 + ym_c^2 + zm_a^2)} + \frac{c^{2(m+1)}}{(xm_c^2 + ym_a^2 + zm_b^2)} \geq \frac{4^{m+1}\sqrt{3}}{3^m(x+y+z)^m}S.$$

Proof: We have:

$$V = \sum_{cyc}\frac{a^{2(m+1)}}{(xm_a^2 + ym_b^2 + zm_c^2)^m} = \sum_{cyc}\frac{(a^2)^{m+1}}{(xm_a^2 + ym_b^2 + zm_c^2)^m},$$

where we apply the inequality of *J. Radon*, and we deduce that:

$$V \geq \frac{(a^2 + b^2 + c^2)^{m+1}}{\left(\sum_{cyc}(xm_a^2 + ym_b^2 + zm_c^2)\right)^m} = \frac{(a^2 + b^2 + c^2)^{m+1}}{(x+y+z)^m(m_a^2 + m_b^2 + c^2)^m}$$

Since,

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2),$$

we obtain that:

$$V \geq \frac{(a^2 + b^2 + c^2)^{m+1}4^m}{3^m(x+y+z)^m(a^2 + b^2 + c^2)^m} = \frac{4^m}{3^m(x+y+z)^m}(a^2 + b^2 + c^2)$$

and by *Ionescu-Weitzenböck*'s inequality, i.e.:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S,$$

we have:

$$V \geq \frac{4^{m+1}\sqrt{3}}{3^m (x+y+z)^m} S,$$

and the proof is complete.

Theorem 3. If $m \in \mathrm{R}_+$, $(x_n)_{n\geq 0}$ is a sequence with $x_0, x_1 \in \mathrm{R}_+^*$, given by $x_{n+2} = x_{n+1} + x_n$, $\forall n \in \mathrm{N}$, then in all triangle ABC, with the area S and usual notations holds the inequality:

$$\frac{a^{2(m+1)}}{\left(x_n m_a^2 + x_{n+1} m_b^2\right)^m} + \frac{b^{2(m+1)}}{\left(x_n m_b^2 + x_{n+1} m_c^2\right)} + \frac{c^{2(m+1)}}{\left(x_n m_c^2 + x_{n+1} m_a^2\right)} \geq \frac{4^{m+1}\sqrt{3}}{3^m x_{n+2}^m} S.$$

Proof: We use the inequality of *J. Radon*, well-known formula

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right)$$

and the inequality of *Ionescu-Weitzenböck*,

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S.$$

We have that:

$$W_n = \sum \frac{a^{2(m+1)}}{\left(x_n m_a^2 + x_{n+1} m_b^2\right)^m} = \sum \frac{\left(a^2\right)^{m+1}}{\left(x_n m_a^2 + x_{n+1} m_b^2\right)^m},$$

and by *J. Radon* ‘s inequality yields that:

$$W_n \geq \frac{\left(a^2 + b^2 + c^2\right)^{m+1}}{\left(\sum\left(x_n m_a^2 + x_{n+1} m_b^2\right)\right)^m} = \frac{\left(a^2 + b^2 + c^2\right)^{m+1}}{\left(x_n + x_{n+1}\right)^m \left(m_a^2 + m_b^2 + m_c^2\right)^m} =$$

$$= \frac{\left(a^2 + b^2 + c^2\right)^{m+1}}{x_{n+2}^m \left(m_a^2 + m_b^2 + m_c^2\right)^m}.$$

Thus,

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right),$$

so,

$$W_n \geq \frac{4^m}{3^m x_{n+2}^m}\left(a^2 + b^2 + c^2\right).$$

and by *Ionescu-Weitzenböck* ‘s inequality, i.e.

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S,$$

we obtain that:

$$W_n \geq \frac{4^{m+1}\sqrt{3}}{3^m x_{n+2}^m} S,$$

and we are done.

Theorem 4. If $m \in R_+$, $(x_n)_{n\geq 0}$ is a sequence with $x_0, x_1 \in R_+^*$, given by $x_{n+2} = x_{n+1} + x_n$, $\forall n \in N$, then in all triangle ABC, with the area S and usual notations holds the inequality:

$$\frac{m_a^{2(m+1)}}{\left(x_n b^2 + x_{n+1}c^2\right)^m} + \frac{m_b^{2(m+1)}}{\left(x_n c^2 + x_{n+1}a^2\right)} + \frac{m_c^{2(m+1)}}{\left(x_n a^2 + x_{n+1}b^2\right)} \geq \frac{3^{m+1}\sqrt{3}}{4^m x_{n+2}^m} S .$$

Proof: We have that

$$Y_n = \sum \frac{m_a^{2(m+1)}}{\left(x_n b^2 + x_{n+1}c^2\right)^m} = \sum \frac{\left(m_a^2\right)^{m+1}}{\left(x_n b^2 + x_{n+1}c^2\right)^m} ,$$

and applying *J. Radon* 's inequality we deduce that:

$$Y_n \geq \frac{\left(m_a^2 + m_b^2 + m_c^2\right)^{m+1}}{\left(\sum\left(x_n b^2 + x_{n+1}c^2\right)\right)^m} = \frac{\left(m_a^2 + m_b^2 + m_c^2\right)^{m+1}}{\left(x_n + x_{n+1}\right)^m\left(a^2 + b^2 + c^2\right)^m} = \frac{\left(m_a^2 + m_b^2 + m_c^2\right)^{m+1}}{x_{n+2}^m\left(a^2 + b^2 + c^2\right)^m} .$$

Because,

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right),$$

hence:

$$Y_n \geq \frac{3^{m+1}}{4^{m+1} x_{n+2}^m}\left(a^2 + b^2 + c^2\right).$$

and by *Ionescu-Weitzenböck*' s inequality, i.e.

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S ,$$

we obtain that:

$$Y_n \geq \frac{3^{m+1}\sqrt{3}}{4^m x_{n+2}^m} S ,$$

i.e. what we must to prove.

Theorem 5. If $m \in R_+$, $(x_n)_{n\geq 0}$ is a sequence with $x_0, x_1 \in R_+^*$, given by $x_{n+2} = x_{n+1} + x_n$, $\forall n \in N$, then in all triangle ABC, with the area S and usual notations holds the inequality:

$$\frac{a^{2(m+1)}}{\left(x_n m_a^2 + x_{n+1}m_b^2 + x_{n+2}m_c^2\right)^m} + \frac{b^{2(m+1)}}{\left(x_n m_b^2 + x_{n+1}m_c^2 + x_{n+2}m_a^2\right)} + \frac{c^{2(m+1)}}{\left(x_n m_c^2 + x_{n+1}m_a^2 + x_{n+2}m_b^2\right)} \geq$$

$$\geq \frac{2^{m+2}\sqrt{3}}{3^m x_{n+2}^m} S .$$

Proof: We have that:

$$Z_n = \sum \frac{a^{2(m+1)}}{\left(x_n m_a^2 + x_{n+1} m_b^2 + x_{n+2} m_c^2\right)^m} = \sum \frac{\left(a^2\right)^{m+1}}{\left(x_n m_a^2 + x_{n+1} m_b^2 + x_{n+2} m_c^2\right)^m},$$

and from *J. Radon*'s inequality follows that:

$$Z_n \geq \frac{\left(a^2+b^2+c^2\right)^{m+1}}{\left(\sum \left(x_n m_a^2 + x_{n+1} m_b^2 + x_{n+2} m_c^2\right)\right)^m} = \frac{\left(a^2+b^2+c^2\right)^{m+1}}{\left(x_n + x_{n+1} + x_{n+2}\right)^m \left(m_a^2+m_b^2+m_c^2\right)^m} =$$
$$= \frac{\left(a^2+b^2+c^2\right)^{m+1}}{2^m x_{n+2}^m \left(m_a^2+m_b^2+m_c^2\right)^m}.$$

Since,

$$m_a^2+m_b^2+m_c^2 = \frac{3}{4}\left(a^2+b^2+c^2\right),$$

we have:

$$Z_n \geq \frac{2^m}{3^m x_{n+2}^m}\left(a^2+b^2+c^2\right).$$

We apply *Ionescu-Weitzenböck*, i.e. $a^2+b^2+c^2 \geq 4\sqrt{3}S$, and we have:

$$Z_n \geq \frac{2^{m+2}\sqrt{3}}{3^m x_{n+2}^m} S, \qquad \text{q.e.d.}$$

Theorem 6. If $m \in \mathrm{R}_+$, $(x_n)_{n\geq 0}$ is a sequence with $x_0, x_1 \in \mathrm{R}_+^*$, given by $x_{n+2} = x_{n+1} + x_n$, $\forall n \in \mathrm{N}$, then in all triangle ABC, with the area S and usual notations holds the inequality:

$$\frac{m_a^{2(m+1)}}{\left(x_n a^2 + x_{n+1} b^2 + x_{n+2} c^2\right)^m} + \frac{m_b^{2(m+1)}}{\left(x_n b^2 + x_{n+1} c^2 + x_{n+2} a^2\right)} + \frac{m_c^{2(m+1)}}{\left(x_n c^2 + x_{n+1} a^2 + x_{n+2} b^2\right)} \geq$$
$$\geq \frac{3^{m+1}\sqrt{3}}{8^m x_{n+2}^m} S.$$

Proof. We have:

$$X_n = \sum \frac{m_a^{2(m+1)}}{\left(x_n a^2 + x_{n+1} b^2 + x_{n+2} c^2\right)^m} = \sum \frac{\left(m_a^2\right)^{m+1}}{\left(x_n a^2 + x_{n+1} b^2 + x_{n+2} c^2\right)^m},$$

and by the inequality of *J. Radon* we obtain that:

$$X_n \geq \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{\left(\sum \left(x_n a^2 + x_{n+1} b^2 + x_{n+2} c^2\right)\right)^m} = \frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{\left(x_n + x_{n+1} + x_{n+2}\right)^m \left(a^2+b^2+c^2\right)^m} =$$

$$=\frac{\left(m_a^2+m_b^2+m_c^2\right)^{m+1}}{2^m x_{n+2}^m\left(a^2+b^2+c^2\right)^m}.$$

From:

$$m_a^2+m_b^2+m_c^2=\frac{3}{4}\left(a^2+b^2+c^2\right),$$

follows

$$X_n\geq\frac{3^{m+1}}{2^{3m+2}x_{n+2}^m}\left(a^2+b^2+c^2\right).$$

Than by the inequality of *Ionescu-Weitzenböck*, i.e.

$$a^2+b^2+c^2\geq 4\sqrt{3}S,$$

yields:

$$X_n\geq\frac{3^{m+1}\sqrt{3}}{2^{3m}x_{n+2}^m}S,$$

and the proofs is complete.

REFERENCES

[1] ***** *Romanian Mathematical Gazette*, **III**(2), 52, 1897.
[2] ***** *Romanian Mathematical Gazette*, **III**(12), 281, 1898.
[3] Weitzenböck, R., Uber eine Ungleichung in der Dreiecksgeometrie, *Mathematische Zeitschrift*, **5**(1-2), 137, 1919.
[4] Engel, A., *Problem solving strategies*, Springer Verlag, 1998.