*ORIGINAL PAPER*

# SOME CONSEQUENCES OF DRIMBE INEQUALITY

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***Abstract.*** *In this paper we present a refinement of inequality:* $\frac{\sum \sin A}{2} \geq \frac{5}{4} + \frac{r}{2R}$ *who are true in any acute triangle ABC and some consequences of DRIMBE inequality.*

***Keywords:*** *concave function, acute triangle, refinement of inequality.*

## MAIN RESULTS

**Theorem 1**. Let $f: I \to R$ be a derivable and convex function on the set I, $f'$ is either convex or concave function on the set I $x, y, z \in I$ . Then:

$$f(x)+f(y)+f(z)+3f\left(\frac{x+y+z}{3}\right) \geq f\left(\frac{2x+y}{3}\right)+f\left(\frac{2y+x}{3}\right)+f\left(\frac{2y+z}{3}\right)+ \\ +f\left(\frac{2z+y}{3}\right)+f\left(\frac{2x+z}{3}\right)+f\left(\frac{2z+x}{3}\right) \geq 2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{z+x}{2}\right)\right] \tag{1}$$

*Proof:* see [1].

**Theorem 2**. Let $f: I \to R$ be a convex function on the set I, $x, y, z \in I$ . Then:

$$f(x)+f(y)+f(z)+f\left(\frac{2x+y}{3}\right)+f\left(\frac{2y+z}{3}\right)+f\left(\frac{2z+x}{3}\right) \geq \\ \geq 2\left[f\left(\frac{x+2y}{3}\right)+f\left(\frac{y+2z}{3}\right)+f\left(\frac{z+2x}{3}\right)\right] \tag{2}$$

*Proof:* see [1].

**Theorem 3**. Let $f: I \to R$ be a convex and derivable function on the set I, $f'$ let be or convex or concave function on the set I, $x, y, z \in I$ , $S \in R^*$. Then:

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$$f(x)+f(y)+f(z)+f\left(\frac{(S-2)x+y+z}{S}\right)+f\left(\frac{(S-2)y+z+x}{S}\right)+f\left(\frac{(S-2)z+x+y}{S}\right)\geq$$
$$\geq 2\left[f\left(\frac{(S-1)x+y}{S}\right)+f\left(\frac{(S-1)x+z}{S}\right)+f\left(\frac{(S-1)y+z}{S}\right)+f\left(\frac{(S-1)y+x}{S}\right)+\right. \quad (3)$$
$$\left.+f\left(\frac{(S-1)z+x}{S}\right)+f\left(\frac{(S-1)z+y}{S}\right)\right]\geq 2\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x+z}{2}\right)+f\left(\frac{z+y}{2}\right)\right]$$

*Proof:* see [1] page 301

If we take S=3 we shall obtain (1)

In the following we shall give a refinement of inequality: $\frac{\sum \sin A}{2}\geq\frac{5}{4}+\frac{r}{2R}$ we are proved in [2], who are true in any acute triangle.

**Corollary 1.1**. In all acute triangle *ABC* holds:

$$\frac{5}{4}+\frac{r}{2R}\leq\frac{1}{2}\left(\frac{\cos(A-B)}{3}+\frac{\cos(B-C)}{3}+\frac{\cos(C-A)}{3}\right)\leq\frac{\sin A}{2}+\frac{\sin B}{2}+\frac{\sin C}{2}$$

*Proof:* In (1) we take: $x=A,\ y=B,\ z=C$ and $f:\left(0,\frac{\pi}{2}\right)\to R$, $f(x)=\cos x$ a concave function with $f'$ a convex function . Because:

$$\cos A+\cos B+\cos C+\frac{3\cos\pi}{3}\leq\frac{\cos(B+2C)}{3}+\frac{\cos(B+2A)}{3}+\frac{\cos(C+2A)}{3}+$$
$$+\frac{\cos(C+2B)}{3}+\frac{\cos(A+2C)}{3}+\frac{\cos(A+2B)}{3}=$$
$$=\cos\left(\frac{\pi}{3}-\frac{A-C}{3}\right)+\cos\left(\frac{\pi}{3}+\frac{A-C}{3}\right)+\cos\left(\frac{\pi}{3}-\frac{A-B}{3}\right)+$$
$$+\cos\left(\frac{\pi}{3}+\frac{A-B}{3}\right)+\cos\left(\frac{\pi}{3}-\frac{B-C}{3}\right)+\cos\left(\frac{\pi}{3}+\frac{B-C}{3}\right)\leq$$
$$\leq 2\left(\frac{\sin A}{2}+\frac{\sin B}{2}+\frac{\sin C}{2}\right)$$

Because $\cos\left(\frac{\pi}{3}+x\right)+\cos\left(\frac{\pi}{3}-x\right)=\cos x$ and $\cos A+coB+\cos C=\frac{R+r}{R}$ it shall result the inequality of the statement.

**Corollary 1.2**. In all triangle *ABC* holds:

$$\sum\cos\alpha\frac{A-B}{3}\geq 3^{1-\alpha}\left(\frac{5}{2}+\frac{r}{R}\right)^{\alpha}\ \text{ for }\ \alpha\geq 1$$

*Proof:* It follows from corollary 1.1. and inequality: $\sum x^{\alpha} \geq 3^{1-\alpha}\left(\sum x\right)^{\alpha}$, for $\alpha \geq 1$

**Corollary 1.3.** In all acute triangle *ABC* holds:

$$\frac{P}{R}+\frac{3\sqrt{3}}{3} \leq \sqrt{3}\left(\frac{\cos(A-B)}{3}+\frac{\cos(B-C)}{3}+\frac{\cos(C-A)}{3}\right) \leq 2\left(\frac{\cos A}{2}+\frac{\cos B}{2}+\frac{\cos C}{2}\right)$$

*Proof:* We shall consider the concave function $f:\left(0,\frac{\pi}{2}\right)\to R$, $f(x)=\sin x$. From (1) it shall result:

$$\sin A+\sin B+\sin C+\frac{3\sin\pi}{3} \leq \sin\left(\frac{\pi}{3}+\frac{A-C}{3}\right)+\sin\left(\frac{\pi}{3}-\frac{A-C}{3}\right)+\sin\left(\frac{\pi}{3}+\frac{A-B}{3}\right)+$$
$$+\sin\left(\frac{\pi}{3}-\frac{A-B}{3}\right)+\sin\left(\frac{\pi}{3}+\frac{B-C}{3}\right)+\sin\left(\frac{\pi}{3}-\frac{B-C}{3}\right) \leq$$
$$\leq 2\left(\frac{\cos A}{2}+\frac{\cos B}{2}+\frac{\cos C}{2}\right)$$

or

$$\frac{P}{R}+\frac{3\sqrt{3}}{3} \leq \sqrt{3}\left(\frac{\cos(A-B)}{3}+\frac{\cos(B-C)}{3}+\frac{\cos(C-A)}{3}\right) \leq 2\left(\frac{\cos A}{2}+\frac{\cos B}{2}+\frac{\cos C}{2}\right)$$

**Corollary 1.4.** In all triangle *ABC* holds:

$$\sum \cos\alpha\frac{A-B}{3} \geq 3\frac{2-3\alpha}{2\left(\frac{P}{R}+\frac{3\sqrt{3}}{2}\right)^{\alpha}},\ \alpha \geq 1$$

*Proof:* It follows from corollary 1.3. and inequality: $\sum x^{\alpha} \geq 3^{1-\alpha}\left(\sum x\right)^{\alpha}$ for $\alpha \geq 1$

**Corollary 1.5**: In all triangle *ABC* holds:

$$\frac{\cos A}{2}+\frac{\cos B}{2}+\frac{\cos C}{2}+\frac{3\sqrt{3}}{2} \leq \sqrt{3}\left(\frac{\cos(A-B)}{6}+\frac{\cos(B-C)}{6}+\frac{\cos(C-A)}{6}\right) \leq$$
$$\leq 2\left(\frac{\cos(\pi-A)}{4}+\frac{\cos(\pi-B)}{4}+\frac{\cos(\pi-C)}{4}\right)$$

*Proof:* It follows from (1) for the concave function: $f:(0,\pi)\to \mathbb{R}$, $f(x)=\frac{\cos x}{2}$ with the $f'$ convex function.

**Remark.** If the angles of a triangle $ABC$ are acute result $\pi - 2A,\ \pi - 2B,\ \pi - 2C$ are angles of a triangle. If we replace in corollary 1.5. we shall obtain the corollary 3 (1.3 ???).

**Corollary 1.6.** In all triangle $ABC$ holds:

$$\frac{\sin A}{2} + \frac{\sin B}{2} + \frac{\sin C}{2} + \frac{3}{2} \le \frac{\cos(A-B)}{6} + \frac{\cos(B-C)}{6} + \frac{\cos(C-A)}{6} \le$$
$$\le 2\left(\frac{\sin(\pi - A)}{4} + \frac{\sin(\pi - B)}{4} + \frac{\sin(\pi - C)}{4}\right)$$

*Proof:* It follows from (1) for $f:(0,\pi) \to \square$, $f(x) = \frac{\sin x}{2}$ concave with $f'$ concave.

**Remark.** For an acute triangle $ABC$ considering $A \to \pi - 2A,\ B \to \pi - 2B,\ C \to \pi - 2C$ we shall obtain the corollary 1.1.

**Corollary 1.7.** In all acute triangle $ABC$ holds:

1) $\sum \cos^{\uparrow} kA + 3/2^{\uparrow} K \ge \sum\left(\square\cos\square^{\uparrow} k(2A+B)/3 + \square\cos\square^{\uparrow} k(2B+A)/3\right) \ge 2\sum \square\sin\square^{\uparrow} kA/2$ if $k < 0$

2) $\sum \square\cos\square^{\uparrow} kA + 3/2^{\uparrow} k \le \sum\left(\square\cos\square^{\uparrow} k(2A+B)/3 + \square\cos\square^{\uparrow} k(2B+A)/3\right) \le 2\sum \square\sin\square^{\uparrow} k\ A/2$ if $k \in \left[\frac{0,2}{3}\right]$

*Proof*. We shall consider the function:

$$f:\left(0,\frac{\pi}{2}\right) \to \square,\ f(x) = \cos^k x,$$
$$f''(x) = -K\cos^{k-2} x\left(1 - k\sin^2 x\right) = -k\cos^{k-2} x\left(\cos^2 x + (1-k)\sin^2 x\right)$$

If $k < 0$, $f$ is a convex function and for $k \in \left[\frac{0,2}{3}\right]$, $f$ is concave.

Also: $(f')'' = k\cos^{k-a} x \sin x\left(3k - 2 - k^2\sin^2 x\right)$. If $k < 0$ $f'$ is a convex function and if $k \in \left[\frac{0,2}{3}\right]$, $f'$ is a concave function .

From (1) it shall result for $k < 0$:

$$\sum \cos^k A + \frac{3}{2^k} \ge \sum\left(\cos^k \frac{2A+B}{3} + \cos^k \frac{2B+A}{3}\right) \ge 2\sum \cos^k \frac{A+B}{2} = 2\sum \sin^k \frac{A}{2}$$

and for $k \in \left[\frac{0,2}{3}\right]$ the reverse of this inequality.

**Corollary 1.8.** In all acute triangle ABC holds:

$$\sum \square\sin\square^{\uparrow} KA+\left(3\left(k+2\right)/2\right)/2^{\uparrow}k \geq$$

1) $$\geq \sum\left[\square\sin\square^{\uparrow} k\left(2A+B\right)/3+\square\sin\square^{\uparrow} k\left(2B+A\right)/3\right]\geq \quad \text{if} \quad k<0$$

$$\geq 2\sum \square\cos\square^{\uparrow} kA/2$$

$$\sum \square\sin\square^{\uparrow} kA+\left(3\left(K+2\right)/2\right)/2^{\uparrow}k \leq$$

2) $$\leq \sum\left[\square\sin\square^{\uparrow} k\left(2A+B\right)/3+\square\sin\square^{\uparrow} K\left(2B+A\right)/3\right]\leq \quad \text{if} \quad k\in\left[\frac{0,2}{3}\right]$$

$$\leq 2\sum \square\cos\square^{\uparrow} kA/2$$

*Proof.* We shall consider the function $f:\left[0,\frac{\pi}{2}\right]\to\square$, $f\left(x\right)=\sin^k x$, $f''\left(x\right)=k\sin^{k-2}x\left[-\sin^2 x+\left(k-1\right)\cos^2 x\right]$.

For $k<0$ $f$ is convex and for $k\in\left[\frac{0,2}{3}\right]$ $f$ is concave.

Also: $\left(f^{\uparrow\prime}\right)^{\uparrow\prime\prime}=k\square\sin\square^{\uparrow}\left(k-3\right)x\cos x\left(-3k+2+k^{\uparrow}2\square\cos\square^{\uparrow}2x\right)$

So, $f$' is concave for $k<0$ and convex for $k<0$ .

Applying inequality (1) for k<0 we obtain:

$$\sum\sin^k A+\left(\frac{\sqrt{3}}{2}\right)^k 3\geq\sum\left[\sin^k\left(\frac{2A+B}{3}\right)+\sin^k\left(\frac{2B+A}{3}\right)\right]\geq 2\sum\sin^k\frac{B+C}{2}$$

or

$$\sum\sin^k A+\frac{3\frac{k+2}{2}}{2^k}\geq\sum\left[\sin^k\frac{2A+B}{3}+\sin^k\frac{2B+A}{3}\right]\geq 2\sum\cos^k\frac{A}{2}$$

For $k\in\left[\frac{0,2}{3}\right]$ we obtain the reverse of the inequality .

**Corollary 1.9.** In all acute triangle *ABC* holds :

1) ???

$$\sum \square\cos\square^{\uparrow} kA+\left(3\left(k+2\right)/2\right)/2^{\uparrow}k \leq$$

2) $$\leq \sum\left(\square\cos\square^{\uparrow} k\left(2A+B\right)/6+\square\cos\square^{\uparrow} k\left(2B+A\right)/6\right)\leq \quad \text{if } k\in\left[\frac{0,2}{3}\right]$$

$$\leq 2\sum \square\cos\square^{\uparrow} k\left(\pi-A\right)/4$$

*Proof.* We shall consider the function

$$f:\left(0,\frac{\pi}{2}\right)\to\square\ ,\ f\left(x\right)=\cos^k\frac{x}{2},$$

$$f''(x) = -\frac{k}{4}\cos^{k-2}\frac{x}{2} + \left(\cos^2\frac{x}{2}(1-k)\sin^2\frac{x}{2}\right)$$

convex for $k<0$ and concave for $k \in \left[\frac{0,2}{3}\right]$ with *f'* convex for $k<0$ and concave for $k \in \left[\frac{0,2}{3}\right]$.

From (1) resulting the requirement.

**Corollary 1.10.** In all acute triangle *ABC* holds:

$$\sum \square\sin\square^{\uparrow} kA/2 + 3/2^{\uparrow} k \ge$$

1) $\ge \sum\left[\square\sin\square^{\uparrow} k(2A+B)/6 + \square\sin\square^{\uparrow} k(2B+A)/6\right] \ge$ if $k<0$

$$\ge 2\sum \square\sin\square^{\uparrow} k(\pi - A)/4$$

$$\sum \square\sin\square^{\uparrow} kA/2 + 3/2^{\uparrow} k \le$$

2) $\le \sum\left[\square\sin\square^{\uparrow} k(2A+B)/6 + \square\sin\square^{\uparrow} K(2B+A)/6\right] \le$ if $k \in \left[\frac{0,2}{3}\right]$

$$\le 2\sum \square\sin\square^{\uparrow} k(\pi - A)/4$$

*Proof.* We shall consider the function: $f:\left(0,\frac{\pi}{2}\right)\to \square$, $f(x) = \sin^k\frac{x}{2}$,
$f''(x) = -\frac{k}{4}\sin^{k-2}\frac{x}{2\left((1-k)^2\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)}$ convex for $k<0$ and concave for $k \in \left[\frac{0,2}{3}\right]$,
*f'* concave for $k<0$ and convex for $k \in \left[\frac{0,2}{3}\right]$.

From inequality (1) inequality follows from the statement.

In the following we give two refinements of the inequalities:

$$\sum ctg\frac{A}{2} \le \frac{pr}{p^2-(2R+r)^2} + \frac{3\sqrt{3}}{2}$$

and

$$\sum tg\frac{A}{2} \le \frac{p^2-4Rr-r^2}{4Rp} + \frac{\sqrt{3}}{2}$$

belonging to [2] who are true in any acute triangle .

**Corollary 1.11.** In all acute triangle *ABC* holds:

1) $\sum ctgA/2 \le 1/2\sum\left(tg(2A+B)/3 + tg(2B+A)/3\right) \le pr/\left(p^{\uparrow}2 - (2R+r)^{\uparrow}2\right) + \left(3\sqrt{3}\right)/2$

2) $\sum\left(tg(2A+B)/3 + tg(2B+A)/3\right) \ge 2p/r$.

*Proof.*

1) Applying to convex function *f*, $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$, $f(x)=tgx$ with the derivate convex the inequality (1) and because $\sum tgA=\frac{2pr}{p^2-(2R+r)^2}$ it follows the inequality of the statement.

2) It follows from point 1) and equality: $\sum ctg\frac{A}{2}=\frac{p}{r}$.

**Corollary 1.12.** In all acute triangle *ABC* holds:

1) $\sum tgA/2 \le 1/2\sum\left(ctg(2A+B)/3+ctg(2B+A)/3\right)\le\left(p\uparrow 2-4Rr-r\uparrow 2\right)/4pr+\sqrt{3}/2$

2) $\sum\left(ctg(2A+B)/3+ctg(2B+A)/3\right)\ge(8R+2r)/p$.

*Proof.*

1) Applying to the convex function *f* with the derivative *f'* concave, $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$, $f(x)=ctgx$ the inequality (1) and from $\sum ctgA=\frac{p^2-4Rr-r^2}{2pr}$ it follows the inequality of the statement.

2) It follows from point 1) and identity: $\sum tg\frac{A}{2}=\frac{4R+r}{p}$.

**Corollary 1.13.** In all acute triangle *ABC* holds:

$$tg^2\frac{\pi-A}{4}tg^2\frac{\pi-B}{4}tg^2\frac{\pi-C}{4}\ge\prod tg\frac{2A+B}{6}\prod tg\frac{2B+A}{6}\ge\frac{\sqrt{3}}{9}\frac{r}{p}.$$

*Proof.* In (1) consider the concave function with derivative *f'* convex $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$, $f(x)=\ln\left(tg\frac{x}{2}\right)$

**Corollary 1.14.** In all triangle *ABC* holds:

$$\frac{\frac{3\sqrt{3}}{16}\frac{pr}{R^2}\le\frac{\prod\sin(2A+B)}{3}\prod\sin(A+2B)}{3}\le\frac{p^2}{16R^2}.$$

*Proof.* We shall consider the concave function with the derivate a convex function $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$, $f(x)=\ln(\sin x)$ in the inequality(1).

**Corollary 1.15.** In all acute triangle *ABC* holds:

$$\frac{\frac{p^2-r^2-4Rr-4R^2}{32R^2}\ge\frac{\prod\cos(2A+B)}{3}\prod\cos(A+2B)}{3}\ge\frac{r^2}{16R^2}$$

*Proof*. We shall consider the function $f:\left(0,\frac{\pi}{2}\right)\to \mathbb{R}$, $f(x)=\ln(\cos x)$ concave with *f'* concave in inequality (1). It shall result:

$$\frac{\prod \cos A\frac{1}{8} \geq \frac{\prod \cos(2A+B)}{3}\prod \cos(A+2B)}{3} \geq \prod \sin^2\frac{A}{2}$$

and because $\prod \cos A=\frac{p^2-r^2-4Rr-4R^2}{4R^2}$ and $\prod \sin^2\frac{A}{2}=\frac{r^2}{16R^2}$. It shall result the required inequality.

**Corollary 1.16.** In all acute triangle *ABC* holds:

$$\sum \frac{1}{\frac{\cos A}{2}} \leq \frac{1}{2}\sum\left(\frac{1}{\frac{\sin(2A+B)}{3}}+\frac{1}{\frac{\sin(A+2B)}{3}}\right) \leq \frac{p^2+r^2+4Rr}{4ep}+\sqrt{3}$$

*Proof*. We shall consider the function $f:\left(0,\frac{\pi}{2}\right)\to \mathbb{R}$, $f(x)=\frac{1}{\sin x}$ convex with *f'* concave. From inequality (1) and identity: $\sum \frac{1}{\sin A}=\frac{p^2+r^2+4Rr}{2pr}$ it follows the inequality of the statement.

**Corollary 1.17.** In all acute triangle *ABC* holds:

$$\sum \frac{1}{\frac{\sin A}{2}} \leq \frac{1}{2}\sum\left(\frac{1}{\frac{\cos(2A+B)}{3}}+\frac{1}{\frac{\cos(A+2B)}{3}}\right) \leq \frac{7p^2-28R^2-24Rr-5r^2}{2\left(p^2-(2R+r)^2\right)}$$

*Proof*. We shall consider the function $f:\left(0,\frac{\pi}{2}\right)\to \mathbb{R}$, $f(x)=\frac{1}{\cos x}$ convex with *f'* convex in the inequality (1).

**Corollary 2.1.** In all triangle holds:

1) $p/R+\sum \sin(2A+B)/3 \leq 2\sum \sin(A+2B)/3$
2) $\sum \cos A/2+\sum \cos(2A+B)/6 \leq 2\sum \cos(A+2B)/6$

*Proof.* 1) From inequality (2) applied to concave function $f:(0,\pi)\to \mathbb{R}$, $f(x)=\sin x$ it shall result:

$$\sum \sin A+\frac{\sum \sin(2A+B)}{3} \leq \frac{2\sum \sin(A+2B)}{3}$$

or

$$\frac{p}{R}+\frac{\sum \sin(2A+B)}{3} \leq \frac{2\sum \sin(A+2B)}{3}$$

2) In (2) we consider the concave function $f:(0,\pi)\to\mathbb{R}$, $f(x)=\frac{\cos x}{2}$.
It shall result: $\frac{\sum\cos A}{2}+\frac{\sum\cos(2A+B)}{6}\le\frac{2\sum\cos(A+2B)}{6}$.

**Corollary 2.2.** In all acute triangle *ABC* holds:

$$1+\frac{r}{R}+\frac{\sum\cos(2A+B)}{3}\le\frac{2\sum\cos(A+2B)}{3}.$$

*Proof.* In (2)we consider the concave function $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$, $f(x)=\cos x$. It shall result:

$$\sum\cos A+\frac{\sum\cos(2A+B)}{3}\le\frac{2\sum\cos(A+2B)}{3}$$

or

$$\frac{R+r}{R}+\frac{\sum\cos(2A+B)}{3}\le\frac{2\sum\cos(A+2B)}{3}$$

**Corollary 2.3.** In all acute triangle *ABC* holds :

1) $\sum\cos^k A+\sum\cos^k(2A+B)/3\ge 2\sum\cos^k(A+2B)/3$ if $k<0$

2) $\sum\cos^k A+\sum\cos^k(2A+B)/3\le 2\sum\cos^k(A+2B)/3$ if $k\in[0,1]$

*Proof.* We shall consider the function

$$f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R},\quad f(x)=\cos^k x,$$

$$f''(x)=-k\cos^{k-2}x\left(1-k\sin^2 x\right)$$

convex for $k<0$ and concave for $k\in[0,1]$ to which we apply (2) .

**Corollary 2.4.** In all triangle *ABC* holds:

1) $\sum\sin^k A+\sum\sin^k(2A+B)/3\ge 2\sum\sin^k(A+2B)/3$ if $k<0$

2) $\sum\sin^k A+\sum\sin^k(2A+B)/3\le 2\sum\sin^k(A+2B)/3$ if $k\in[0,1]$

*Proof.* We shall consider the function: $f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}$, $f(x)=\sin^k x$, $f''(x)=k\sin^{k-2}x\left[-\sin^2 x+(k-1)\cos^2 x\right]$ convex for $k<0$ and concave for $k\in[0,1]$ to which we apply (2)

**Corollary 2.5.** In all acute triangle ABC holds:

1) $\sum\sin^k 2A+\sum\sin^k(4A+2B)/3\ge 2\sum\sin^k(2A+4B)/3$ if $k<0$

2) $\sum\sin^k 2A+\sum\sin^k(4A+2B)/3\le 2\sum\sin^k(2A+4B)/3$ if $k\in[0,1]$

*Proof*. In corollary 2.4 we shall consider: $A \to \pi - 2A$, $B \to \pi - 2B$, $C \to \pi - 2C$.

**Corollary 2.6.** In all triangle $ABC$ holds:

1) $\sum \cos kA/2 + \sum \cos k(2A+B)/6 \ge 2\sum \cos k(A+2B)/6$ if $k < 0$

2) $\sum \cos kA/2 + \sum \cos k(2A+B)/6 \le 2\sum \cos k(A+2B)/6$ if $k \in [0,1]$

*Proof*. In corollary 2.4 we shall consider: $A \to \frac{\pi - A}{2}$, $B \to \frac{\pi - B}{2}$, $C \to \frac{\pi - C}{2}$.

**Corollary 2.7.** In all acute triangle $ABC$ holds:

$$\frac{2r}{R^2}p + \frac{\sum \sin(4A+2B)}{3} \le \frac{2\sum \sin(2A+4B)}{3}$$

*Proof*. From point 2) corollary 2.5. for k = 1 result: $\sum \sin 2A + \frac{\sum \sin(4A+2B)}{3} \le \frac{2\sum \sin(2A+4B)}{3}$ and because $\sum \sin 2A = \frac{2r}{R^2}p$ we have $\frac{2r}{R^2}p + \frac{\sum \sin(4A+2B)}{3} \le \frac{2\sum \sin(2A+4B)}{3}$.

**Corollary 3.1.** In all acute triangle $ABC$ holds:

$$\frac{\frac{p}{R} + \frac{\sum \sin(\pi + (S-3)A)}{S} \le \frac{2\sum \cos C}{2}\cos((S-2))}{2S}(A-B) \le \frac{2\sum \cos A}{2}, \quad S \in \mathbb{N}^*$$

*Proof*. From inequality (3) apply to function: $f : \left(0, \frac{\pi}{2}\right) \to \mathbb{R}$ concave with concave derivate: $f(x) = \sin x$ result:

$$\sum \sin A + \frac{\sum \sin((S-3)A+\pi)}{S} \le \sum\left(\frac{\sin((S-1)A+B)}{S} + \frac{\sin((S-1)B+A)}{S}\right) \le \frac{2\sum \sin(B+C)}{2}$$

equivalent with:

$$\frac{\frac{p}{R} + \frac{\sum \sin(\pi + (S-3)A)}{S} \le \frac{2\sum \cos C}{2}\cos(S-2)(A-B)}{2S} \le \frac{2\sum \cos A}{2}$$

**Corollary 3.2.** In all acute triangle $ABC$ holds:

$$\frac{\frac{R+r}{R} + \frac{\sum \cos((S-3)A+\pi)}{S} \le \frac{2\sum \sin C}{2}\cos(S-2)(A-B)}{2S} \le \frac{2\sum \sin A}{2}, \quad S \in \mathbb{N}^*$$

*Proof.* From inequality (3) applied to concave function with *f'* convex $f:\left(0,\frac{\pi}{2}\right)\to \mathbb{R}$, $f(x)=\cos x$ result

$$\sum\cos A+\frac{\sum\cos\left(\left(S-3\right)A+\pi\right)}{S}\le\sum\left(\frac{\cos\left(\left(S-1\right)A+B\right)}{S}+\frac{\cos\left(\left(S-1\right)B+A\right)}{S}\right)\le\frac{2\sum\sin A}{2}$$

equivalent with:

$$\frac{\frac{R+r}{R}+\frac{\sum\cos\left(\left(S-3\right)A+\pi\right)}{S}\le\frac{2\sum\sin C}{2}\cos\left(\left(S-2\right)\right)}{2S}\left(A-B\right)\le\frac{2\sum\sin A}{2}$$

**Corollary 3.3**. In all triangle *ABC* holds:

$$\sum\cos A/2+\frac{\sum\cos\left(\pi+\left(S-3\right)A-3\right)}{2S}\le$$
$$\le 2\cos\left(\pi-C\right)/4\cos\left(\left(S-2\right)A-B\right)/4S\le \quad , S\in\mathbb{N}^*$$
$$\le\sum\cos\left(\pi-A\right)/4$$

*Proof.* In (3) consider the concave function *f* and *f'* convex function $f:\left(0,\pi\right)\to\mathbb{R}$, $f(x)=\frac{\cos x}{2}$. It shall result:

$$\frac{\sum\cos A}{2}+\frac{\sum\cos\left(\left(S-2\right)A+B-C\right)}{2S}\le\sum\left(\frac{\cos\left(\left(S-1\right)A+B\right)}{2S}+\frac{\cos\left(\left(S-1\right)B+A\right)}{2S}\right)\le$$
$$\le\frac{2\sum\cos\left(B+C\right)}{4}$$

it follows the inequality from the statement.

**Corollary 3.4**. In all triangle *ABC* holds:

$$\frac{\frac{\sum\sin A}{2}+\frac{\sum\sin\left(\pi+\left(S-3\right)A\right)}{2S}\le\frac{2\sum\sin\left(\pi-A\right)}{4}\cos\left(S-2\right)\left(A-B\right)}{4S}\le \quad , S\in\mathbb{N}^*$$
$$\le\frac{2\sum\sin\left(\pi-A\right)}{4}$$

*Proof.* In (3) we shall consider the concave function with *f'* concave $f:\left(0,\pi\right)\to\mathbb{R}$, $f(x)=\frac{\sin x}{2}$.

It shall result:

$$\frac{\sum\sin A}{2}+\frac{\sum\sin\left(\left(S-2\right)A+B+C\right)}{2S}\le\sum\left(\frac{\sin\left(\left(S-1\right)A+B\right)}{2S}+\frac{\sin\left(\left(S-1\right)B+A\right)}{2S}\right)\le$$
$$\le\frac{2\sum\sin\left(\pi-A\right)}{4}$$

**Corollary 3.5.** In all acute triangle *ABC* holds:

1)
$$\sum \square\cos\square^{\uparrow} kA + \sum \square\cos\square^{\uparrow} k\left(\pi + (S-3)A\right)/S \ge$$
$$\ge \sum \square\cos\square^{\uparrow} k\left((S-1)A + B\right)/S + \square\cos\square^{\uparrow} k\left((S-1)B + A\right)/S \ge \quad \text{if} \quad k < 0,\ S \in \square\,*$$
$$\ge 2\sum \square\sin\square^{\uparrow} kA/2$$

2)
$$\sum \square\cos\square^{\uparrow} kA + \sum \square\cos\square^{\uparrow} k\left(\pi + (S-3)A\right)/S \le$$
$$\le \sum \square\cos\square^{\uparrow} k\left((S-1)A + B\right)/S + \square\cos\square^{\uparrow} k\left((S-1)B + A\right)/S \le \quad \text{if} \quad k \in \left[\frac{0,2}{3}\right],\ S \in \square\,*$$
$$\le 2\sum \square\sin\square^{\uparrow} kA/2$$

*Proof.* We shall consider the function $f : \left(0, \frac{\pi}{2}\right) \to \square$, $f(x) = \cos^k x$ convex for $k < 0$ and concave for $k \in \left[\frac{0,2}{3}\right]$ with *f'* convex for $k < 0$ and concave for $k \in \left[\frac{0,2}{3}\right]$ to which we apply (3)

**Corollary 3.6**. In all acute triangle *ABC* holds:

1)
$$\sum \square\sin\square^{\uparrow} kA + \sum \square\sin\square^{\uparrow} k\left(\pi + (S-3)A\right)/S \ge$$
$$\ge \sum \square\sin\square^{\uparrow} k\left((S-1)A + B\right)/S + \square\sin\square^{\uparrow} k\left((S-1)B + A\right)/S \ge \quad \text{if} \quad k < 0,\ S \in \square\,*$$
$$\ge 2\sum \square\cos\square^{\uparrow} kA/2$$

2)
$$\sum \square\sin\square^{\uparrow} kA + \sum \square\sin\square^{\uparrow} k\left(\pi + (S-3)A\right)/S \le$$
$$\le \sum \square\sin\square^{\uparrow} k\left((S-1)A + B\right)/S + \square\sin\square^{\uparrow} k\left((S-1)B + A\right)/S \le \quad \text{if} \quad k \in \left[\frac{0,2}{3}\right],\ S \in \square\,*$$
$$\le 2\sum \square\cos\square^{\uparrow} kA/2$$

*Proof.* We shall consider the function $f : \left(0, \frac{\pi}{2}\right) \to \square$, $f(x) = \sin^k x$ convex for $k < 0$ and concave for $k \in \left[\frac{0,2}{3}\right]$ with *f'* concave for $k < 0$ and convex for $k \in \left[\frac{0,2}{3}\right]$ to which we apply (3).

## REFERENCES


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