***ORIGINAL PAPER***

# SOME GENERALIZATIONS OF IONESCU-WEITZENBÖCK 'S INEQUALITY*

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*Manuscript received: 11.01.2013; Accepted paper: 03.02.2013;*

*Published online: 01.03.2013.*

***Abstract.*** *In this paper we give some generalizations of the inequality of Ionescu-Weitzenböck.*

***Keywords:*** *Ionescu-Weitzenböck's inequality, Bergstrom's inequality, Jensen's inequality, Mitrinović's inequality, Jakob Steiner's theorem, Radon's inequality.*

## 1. INTRODUCTION

In [1] we give 23 proofs of the inequality of *Ionescu-Weitzenböck*, i.e.:
In all triangle $ABC$ with usual notations holds:

$$a^2+b^2+c^2 \geq 4S\sqrt{3} \qquad \text{(I-W)}$$

Here we give 10 generalizations of (I-W) as follows:

## 2. MAIN RESULTS

**Generalization 1.** If $m \in [1,\infty)$, then all triangle $ABC$, holds the inequality:

$$a^{2m}+b^{2m}+c^{2m} \geq 3\cdot\left(\frac{4S}{\sqrt{3}}\right)^m .$$

*Proof 1.* We have:

$$a^{2m}+b^{2m}+c^{2m}=(a^2)^m+(b^2)^m+(c^2)^m,$$

where we taking account that the function

$$u: R_+^* \to R_+^*,\ u(x)=x^m$$

is increasing and convex on $R_+^*$, and applying *Jensen*'s inequality we obtain that:

$$u(a^2)+u(b^2)+u(c^2) \geq 3u\left(\frac{a^2+b^2+c^2}{3}\right) \geq 3u\left(\frac{4\sqrt{3}S}{3}\right)=3u\left(\frac{4S}{\sqrt{3}}\right) \Leftrightarrow$$

$$\Leftrightarrow a^{2m}+b^{2m}+c^{2m} \geq 3\left(\frac{a^2+b^2+c^2}{3}\right)^m \geq 3\left(\frac{4S}{\sqrt{3}}\right)^m .$$

*Proof 2.* Since the function $f: R_+^* \to R_+^*,\ f(x)=x^{2m}$, is convex on $R_+^*$ by *Jensen*'s inequality we deduce that:

**** Paper in memoriam of Ion Ionescu (1870-1946).***

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$$f(a)+f(b)+f(c)\ge\frac{1}{3}f\left(\frac{a+b+c}{3}\right)\Leftrightarrow a^{2m}+b^{2m}+c^{2m}\ge\frac{1}{3^{2m-1}}(2p)^{2m}=$$

$$=\frac{4^m p^{2m}}{3^{2m-1}}=\frac{4^m p^m r^m p^m}{3^{2m-1}r^m}=\frac{4^m S^m}{3^{2m-1}}\left(\frac{p}{r}\right)^m.$$

where we use *Mitrinović*'s inequality, i.e. $\frac{p}{r}\ge 3\sqrt{3}$, and yields the result.

*Proof 3.* From *J. Radon*'s inequality we have:

$$a^{2m}+b^{2m}+c^{2m}\ge\frac{(a+b+c)^{2m}}{3}=\frac{4^m p^{2m}}{3^{2m-1}},$$

then proceed as in the Proof 2.

**Observation 1.** If $m=1$, then by the generalization 1 we obtain the inequality of *Ionescu-Weitzenböck*.

**Generalization 2.** If $m\in[2,\infty)$, then in all triangle $ABC$, holds:

$$\frac{a^m+b^m+c^m}{3}\ge\left(\frac{4S}{\sqrt{3}}\right)^{\frac{m}{2}}.$$

*Proof.* We have:

$$a^m+b^m+c^m=(a^2)^{\frac{m}{2}}+(b^2)^{\frac{m}{2}}+(c^2)^{\frac{m}{2}},$$

and since the function $v:R_+^*\to R_+^*,\ v(x)=x^{\frac{m}{2}}$ is increasing and convex on $R_+^*$, we apply the inequality of *Jensen* and we deduce that:

$$v(a^2)+v(b^2)+v(c^2)\ge 3v\left(\frac{a^2+b^2+c^2}{3}\right)\ge 3v\left(\frac{4\sqrt{3}S}{3}\right)=3v\left(\frac{4S}{\sqrt{3}}\right)\Leftrightarrow$$

$$\Leftrightarrow a^m+b^m+c^m\ge 3\left(\frac{4S}{\sqrt{3}}\right)^{\frac{m}{2}}.$$

**Observation 2.** If $m=2$, then by the generalization 2 we obtain the inequality of *Ionescu-Weitzenböck*.

**Generalization 3.** If the function $w:R_+^*\to R_+^*$ is convex and increasing on $R_+^*$, then in all triangle $ABC$, holds:

$$w(a^2)+w(b^2)+w(c^2)\ge 3w\left(\frac{4S}{\sqrt{3}}\right).$$

*Proof.* By *Jensen*'s inequality we have:

$$w(a^2)+w(b^2)+w(c^2)\ge 3w\left(\frac{a^2+b^2+c^2}{3}\right)$$

Where we taking account (I-W) and the fact that $w$ is increasing on $R_+^*$ we obtain that:

$$w(a^2)+w(b^2)+w(c^2)\ge 3w\left(\frac{4S\sqrt{3}}{3}\right)=3w\left(\frac{4S}{\sqrt{3}}\right).$$

**Observation 3.** If $w(x) = x$, then by the generalization 3 we obtain the inequality of *Ionescu-Weitzenböck*.

**Generalization 4.** If $m, x, y, z \in R_+$ şi $x + y + z \in R_+^*$, then in all triangle $ABC$, holds:

$$\frac{a^{m+2}}{(xa + yb + zc)^m} + \frac{b^{m+2}}{(xb + yc + za)^m} + \frac{c^{m+2}}{(xc + ya + zb)^m} \geq \frac{4\sqrt{3}S}{(x + y + z)^m}.$$

*Proof.* We have:

$$U = \frac{a^{m+2}}{(xa + yb + zc)^m} + \frac{b^{m+2}}{(xb + yc + za)^m} + \frac{c^{m+2}}{(xc + ya + zb)^m} =$$

$$= \frac{a^{2(m+1)}}{(xa^2 + yab + zac)^m} + \frac{b^{2(m+1)}}{(xb^2 + ybc + zab)^m} + \frac{c^{2(m+1)}}{(xc^2 + yac + zbc)^m} =$$

$$= \frac{(a^2)^{m+1}}{\left(xa^2 + \frac{y}{2}(a^2 + b^2) + \frac{z}{2}(a^2 + c^2)\right)^m} + \frac{(b^2)^{m+1}}{\left(xb^2 + \frac{y}{2}(b^2 + c^2) + \frac{z}{2}(a^2 + b^2)\right)^m} +$$

$$+ \frac{(c^2)^{m+1}}{\left(xc^2 + \frac{y}{2}(a^2 + c^2) + \frac{z}{2}(b^2 + c^2)\right)^m},$$

where we apply *J. Radon*'s inequality and (I-W) and we obtain:

$$U \geq \frac{\left(a^2 + b^2 + c^2\right)^{m+1}}{\left(x(a^2 + b^2 + c^2) + y(a^2 + b^2 + c^2) + z(a^2 + b^2 + c^2)\right)^m} = \frac{a^2 + b^2 + c^2}{(x + y + z)^m} \geq$$

$$\geq \frac{4\sqrt{3}S}{(x + y + z)^m}.$$

**Observation 4.** If $m = 0$, then by the generalization 4 we obtain the inequality of *Ionescu-Weitzenböck*.

**Generalization 5.** For any point $M \in IntABC$ we denote $d_a(M), d_b(M), d_c(M)$ the distances from the point $M$ to the lines $BC, CA, AB$ and $s_a(M) = Area[MBC]$, $s_b(M) = Area[MCA]$, $s_c(M) = Area[MAB]$.

If $m \in R_+$, then in all triangle $ABC$ holds:

$$\frac{a^{m+2}}{d_a^m(M)} + \frac{b^{m+2}}{d_b^m(M)} + \frac{c^{m+2}}{d_c^m(M)} \geq 2^{m+2}(\sqrt{3})^{m+1} S.$$

*Proof.* We have:

$$U = \sum \frac{a^{m+2}}{d_a^m(M)} = \sum \frac{a^{2(m+1)}}{a^m d_a^m(M)} = \sum \frac{(a^2)^{m+1}}{2^m s_a^m(M)},$$

and by the inequality of *J. Radon* we deduce that :

$$U \geq \frac{\left(a^2+b^2+c^2\right)^{m+1}}{2^m\left(s_a(M)+s_b(M)+s_c(M)\right)^m}=\frac{\left(a^2+b^2+c^2\right)^{m+1}}{2^m S^m},$$

where we use the fact that:

$$a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3}=\frac{4p^2}{3},$$

therefore:

$$U \geq \frac{4^{m+1}p^{2(m+1)}}{2^m 3^{m+1} S^m} \geq \frac{2^{m+2}p^{m+1}\left(3\sqrt{3}r\right)^{m+1}}{3^{m+1}}=\frac{2^{m+2}\left(\sqrt{3}\right)^{m+1}(pr)^{m+1}}{S^m}=\frac{2^{m+2}\left(\sqrt{3}\right)^{m+1}S^{m+1}}{S^m}=$$
$$=2^{m+2}(\sqrt{3})^{m+1}S.$$

**Observation 5.** If $m=0$, then by the generalization 5 we obtain the inequality of *Ionescu-Weitzenböck*.

**Generalization 6.** If $m \in R_+^*$, then in all triangle $ABC$ holds:

$$\left(\frac{a^m+b^m+c^m}{3}\right)^{\frac{2}{m}} \geq \frac{4\sqrt{3}}{3}S.$$

*Proof.* We have:

$$S=\frac{bc\sin A}{2}=\frac{ca\sin B}{2}=\frac{ab\sin C}{2},$$

So

$$S^3=\frac{(abc)^2\sin A\sin B\sin C}{8}.$$

Since,

$$\sin A+\sin B+\sin C \geq 3\sqrt[3]{\sin A\sin B\sin C},$$

yields that :

$$S^3 \leq \frac{a^2b^2c^2}{8}\left(\frac{\sin A+\sin B+\sin C}{3}\right)^3,$$

where we use the fact:

$$\sin A+\sin B+\sin C \leq \frac{3\sqrt{3}}{2},$$

and we get:

$$S^3 \leq \frac{a^2b^2c^2}{8}\left(\frac{\sqrt{3}}{2}\right)^3 \Leftrightarrow (abc)^2 \geq \left(\frac{4S}{\sqrt{3}}\right)^3.$$

Therefore:

$$\frac{a^m+b^m+c^m}{3} \geq \frac{3\sqrt[3]{a^m b^m c^m}}{3}=(abc)^{\frac{m}{3}} \Leftrightarrow$$
$$\Leftrightarrow \left(\frac{a^m+b^m+c^m}{3}\right)^{\frac{2}{m}} \geq (abc)^{\frac{2}{3}}=\left((abc)^2\right)^{\frac{1}{3}} \geq \frac{4S}{\sqrt{3}}.$$

**Observation 6.** If $m=2$, then by the generalization 6 we obtain the inequality of *Ionescu-Weitzenböck*.

**Generalization 7.** If $x,y\in R_+$, $x+y\in R_+^*$, then in any triangle $ABC$, holds:

$$(xa+yb)^2+(xb+yc)^2+(xc+ya)^2\geq 4\sqrt{3}(x+y)^2 S.$$

*Proof.* By the inequality of *H. Bergström* we have that:

$$U=(xa+yb)^2+(xb+yc)^2+(xc+ya)^2\geq\frac{(x+y)^2(a+b+c)^2}{3}=\frac{4p^2(x+y)^2}{3}.$$

By the theorem of *Jakob Steiner*, i.e if we consider all triangles with the perimeter $2p$, then the triangle with the maximum area is equilateral triangle. Let $S_3$ be the area of the equilateral triangle with the perimeter $2p$ and $a=\frac{2p}{3}$ be the lenght of the side. We have:

$$S_3=\frac{a^2\sqrt{3}}{4}=\frac{4p^2}{9}\cdot\frac{\sqrt{3}}{4}=\frac{p^2\sqrt{3}}{9}\Leftrightarrow 9S_3=p^2\sqrt{3}\Leftrightarrow p^2=3\sqrt{3}S_3.$$

Hence,

$$U\geq\frac{(x+y)^2(a+b+c)^2}{3}=\frac{4p^2(x+y)^2}{3}=4\sqrt{3}(x+y)^2S_3,$$

and because $S_3\geq S$ (by the theorem of *Jakob Steiner*) we get:

$$U\geq 4\sqrt{3}(x+y)^2 S.$$

**Observation 7.** If $x=1, y=0$, then by the generalization 7 we obtain the inequality of *Ionescu-Weitzenböck*.

**Generalization 8.** If $m,x,y\in R_+$, $x+y\in R_+^*$, then in all triangle $ABC$, holds:

$$(xa+yb)^{m+1}+(xb+yc)^{m+1}+(xc+ya)^{m+1}\geq 2^{m+1}(x+y)^{m+1}3^{\frac{3-m}{4}}S^{\frac{m+1}{2}}.$$

*Proof.* By J. Radon's inequality we have that:

$$V=(xa+yb)^{m+1}+(xb+yc)^{m+1}+(xc+ya)^{m+1}\geq\frac{(x+y)^{m+1}}{3^m}(a+b+c)^{m+1}=$$

$$=\frac{(x+y)^{m+1}}{3^m}2^{m+1}p^{m+1}=\frac{(x+y)^{m+1}}{3^m}2^{m+1}(p^2)^{\frac{m+1}{2}}.$$

As in the generalization 7 we have: $p^2=3\sqrt{3}S_3$ and by *Jakob Steiner*'s theorem we have $S_3\geq S$.

So,

$$V\geq\frac{(x+y)^{m+1}}{3^m}2^{m+1}\left(p^2\right)^{\frac{m+1}{2}}=\frac{(x+y)^{m+1}}{3^m}2^{m+1}\left(3\sqrt{3}S_3\right)^{\frac{m+1}{2}}\geq$$

$$\geq\frac{(x+y)^{m+1}}{3^m}2^{m+1}\left(3\sqrt{3}S\right)^{\frac{m+1}{2}}=2^{m+1}(x+y)^{m+1}3^{\frac{3-m}{4}}S^{\frac{m+1}{2}}$$

, q.e.d.

**Observation 8.** If $m=x=1, y=0$, then by the generalization 8 we obtain the inequality of *Ionescu-Weitzenböck*.

**Generalization 9.** If $x, y, z \in R_+, x+y+z \in R_+^*$, then in all triangle $ABC$, holds:

$$(xa+yb+zc)^2+(xb+yc+za)^2+(xc+ya+zb)^2 \geq 4\sqrt{3}(x+y+z)^2 S.$$

*Proof.* By *H. Bergström*'s inequality we have that:

$$U=(xa+yb+zc)^2+(xb+yc+za)^2+(xc+ya+zb)^2 \geq \frac{(x+y+z)^2(a+b+c)^2}{3}=$$

$$=\frac{4p^2(x+y+z)^2}{3}.$$

Like above we have: $p^2=3\sqrt{3}S_3$ and $S_3 \geq S$.

Hence,

$$U \geq \frac{4(x+y+z)^2}{3}3\sqrt{3}S=4\sqrt{3}(x+y+z)^2 S.$$

**Observation 9.** If $x=1, y=z=0$, then by the generalization 9 we obtain the inequality of *Ionescu-Weitzenböck*.

**Generalization 10.** If $m, x, y, z \in R_+, x+y+z \in R_+^*$, then in all triangle $ABC$, holds:

$$(xa+yb+zc)^{m+1}+(xb+yc+za)^{m+1}+(xc+ya+zb)^{m+1} \geq 2^{m+1}(x+y+z)^{m+1}3^{\frac{3-m}{4}}S^{\frac{m+1}{2}}.$$

*Proof.* By *J. Radon*'s inequality we have that:

$$V=\sum(xa+yb+zc)^{m+1} \geq \frac{\left(\sum(xa+yb+zc)\right)^{m+1}}{3^m}=$$

$$=\frac{(x+y+z)^{m+1}(a+b+c)^{m+1}}{3^m}=\frac{(x+y+z)^{m+1}}{3^m}2^{m+1}p^{m+1}.$$

As above: $p^2=3\sqrt{3}S_3$ and $S_3 \geq S$.

Thus,

$$V \geq \frac{(x+y+z)^{m+1}2^{m+1}}{3^m}(p^2)^{\frac{m+1}{2}} \geq \frac{(x+y+z)^{m+1}2^{m+1}}{3^m}(3\sqrt{3}S)^{\frac{m+1}{2}}=$$

$$=2^{m+1}(x+y+z)^{m+1}3^{\frac{3-m}{4}}S^{\frac{m+1}{2}}, \text{ q.e.d.}$$

**Observation 10.** If $m=x=1, y=z=0$, then by the generalization 10 we obtain the inequality of *Ionescu-Weitzenböck*.

## REFERENCE

[1] Bătineţu-Giurgiu, D.M., Stanciu, N., *Gaz. Mat.*-B, **118**(1), 1, 2013.