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A NEW INEQUALITY AND IDENTITY

DAM VAN NHI[1]

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Abstract. *In this paper we introduce the new inequality and identity called (M, N), that Hayashi's inequality is only a special case. Then we will present some interesting applications.*

Keywords: *Hayashi's inequality, triangle, polygon.*

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1. INTRODUCTION

Suppose given a triangle ΔABC of the lengths of sides *a, b, c*. Hayashi propose an inequality: With any point *M*, we have

$$aMB.MC + bMC.MA + cMA.MB \geq abc$$

(see [2, 3]). In this paper we propose a new inequality which is a generalization of the Hayashi's inequality, then we present some interesting applications in triangle. Successfully, we have two following principal results.

Theorem 1.1. Let $A_1A_2...A_n$ be a polygon, *s* be an integer, $s < n$, and arbitrary points $N_1, N_2, ..., N_s,\ M$ in euclidean plane $\mathbb{R}^2$ we have the following inequality

$$\frac{\prod_{j=1}^{s} MN_j}{\prod_{i=1}^{n} MA_i} \leq \sum_{k=1}^{n} \frac{\prod_{j=1}^{s} A_kN_j}{\prod_{i\neq k} A_kA_i.MA_k}$$

We call this inequality as name the inequality(M, N).

(i) If $s = 0$, we have Hayashi's inequality.

(ii) If $n = 3,\ s = 1$, and *A, B, C, N* belong to the circle with the center *M* we have the inequality $aAN + bBN + cCN \geq 4S_{ABC}$.

[1] School for gifted students, Hanoi National University of Education, Hanoi, Vietnam.
Email: damvannhi@yahoo.com.

Proposition 1.2. Assume that the polygon $A_1A_2...A_n$ is inscribed in the circle with the center *O* and radius *R*. Taking $s+1$ points $N_1, N_2,..., N_s$ and *M* also belonging to this circle *C*. Assuming that the coordinate $A_k\left(\cos\alpha_k;\sin\alpha_k\right)$, $k=1,2,...,n$; the coordinate $N_j\left(\cos u_j;\sin u_j\right)$, $j=1,2,...,s$ and the coordinate $M\left(\cos u;\sin u\right)$. Then, we will have these identities

(i) $$\frac{\prod_{j=1}^{s}\sin\frac{u-u_j}{2}}{\prod_{t=1}^{n}\sin\frac{u-\alpha_t}{2}}=\sum_{k=1}^{n}\frac{\prod_{j=1}^{s}\sin\frac{\alpha_k-u_j}{2}}{\sin\frac{u-\alpha_k}{2}\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}}\cos\frac{(s+1-n)(\alpha_k-u)}{2}$$

(ii) $$\sum_{k=1}^{n}\frac{\prod_{j=1}^{s}\sin\frac{\alpha_k-u_j}{2}}{\sin\frac{u-\alpha_k}{2}\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}}\sin\frac{(s+1-n)(\alpha_k-u)}{2}=0$$

(iii) $$\sum_{k=1}^{3}\frac{\sin\frac{\alpha_k-u_1}{2}}{\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}}=0 \text{ if } n=3,\ s=1$$

(iv) $$\frac{\prod_{j=1}^{n-1}\sin\frac{u-u_j}{2}}{\prod_{t=1}^{n}\sin\frac{u-\alpha_t}{2}}=\sum_{k=1}^{n}\frac{\prod_{j=1}^{n-1}\sin\frac{\alpha_k-u_j}{2}}{\sin\frac{u-\alpha_k}{2}\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}} \text{ if } s=n-1$$

(v) $$\frac{\prod_{j=1}^{n-2}\sin\frac{u_j}{2}}{\prod_{t=1}^{n}\sin\frac{\alpha_t}{2}}=\sum_{k=1}^{n}\frac{\prod_{j=1}^{n-2}\sin\frac{\alpha_k-u_j}{2}}{\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}}\cot\frac{\alpha_t}{2} \text{ and } \sum_{k=1}^{n}\frac{\prod_{j=1}^{n-2}\sin\frac{\alpha_k-u_j}{2}}{\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}}=0 \text{ if } s=n-2$$

2. INEQUALITY AND IDENTITY

Now we prove an inequality that Hayashi's inequality is a special case.

Theorem 2.1. Let $A_1A_2...A_n$ be a polygon, *s* be an integer, $s<n$, and arbitrary points $N_1, N_2,..., N_s,\ M$ in Euclidean plane $\mathbb{R}^2$ we have the following inequality

$$\frac{\prod_{j=1}^{s} MN_j}{\prod_{i=1}^{n} MA_i} \le \sum_{k=1}^{n} \frac{\prod_{j=1}^{s} A_k N_j}{\prod_{i\ne k} A_k A_i . MA_k}$$

We call this inequality as name the inequality (M, N).

(i) If $s = 0$, we have Hayashi's inequality.

(ii) If $n = 3,\ s = 2$, and *A, B, C, N* belong to the circle with the center *M* we have the inequality $aAN + bBN + cCN \ge 4S_{ABC}$.

Proof: Suppose that A_k have affixe a_k, M has affixe *z* and N_h affixe z_h. Using the Lagrange interpolation formula, we have $\prod_{j=1}^{s}(z - z_j) = \sum_{k=1}^{n} \frac{\prod_{j=1}^{s}(a_k - z_j)}{\prod_{i\ne k}(a_k - a_i)} \prod_{i\ne k}(z - a_i)$ and deducing $\frac{\prod_{j=1}^{s}|z - z_j|}{\prod_{i=1}^{n}|z - a_i|} \le \sum_{k=1}^{n} \frac{\prod_{j=1}^{s}|a_k - z_j|}{\prod_{i\ne k}|a_k - a_i||z - a_k|}$. From this, we deduce the geometric inequality

$$\frac{\prod_{j=1}^{s} MN_j}{\prod_{i=1}^{n} MA_i} \le \sum_{k=1}^{n} \frac{\prod_{j=1}^{s} A_k N_j}{\prod_{i\ne k} A_k A_i . MA_k}$$

(i) If $s = 0$ we have $\prod MN_j = 1 = \prod A_k N_j$ and the inequality (M, N) becomes the Hayashi's inequality for the polygon

$$\frac{1}{\prod_{i=1}^{n} MA_i} \le \sum_{k=1}^{n} \frac{\prod_{j=1}^{s} A_k N_j}{\prod_{i\ne k} A_k A_i . MA_k}$$

(ii) If $n = 3,\ s = 1$ and *A, B, C, N* belong to the circle with the center *M* we have the inequality $\frac{abc}{R} \le aAn + bBN + cCN$ or $aAN + bBN + cCN \ge 4S_{ABC}$. □

Remark 2.2. Denote *N* as the center of circumcircle. Applying the inequality (M, N) (ii) we deduce $R(a + b + c) \ge 2r(a + b + c)$ or $R \ge 2r$ [**Euler**].

Corollary 2.3. Suppose that *O, I* and *G* are respectively the center of circumcirle and incircle of ΔABC. Denote the radii of circumcircles of the triangles *GBC, GCA, GAB* by R_1, R_2, R_3, respectivly. Let r_a, r_b, r_c be the radii of circumcircle of the triangles *IBC, ICA, IAB,* and let $R_1^{'}$, $R_2^{'}$, $R_3^{'}$ be the radii of circumcircles of the triangles *OBC, OCA, OAB*, respectively. We have

(i) $R^2 \geq \dfrac{abc}{a+b+c}$

(ii) $R_1 + R_2 + R_3 \geq 3R$ (see [1]).

(iii) $\dfrac{r_a}{h_a} + \dfrac{r_b}{h_b} + \dfrac{r_c}{h_c} \geq \dfrac{R}{r}$ where h_a, h_b, h_c are the lengths of altitudes of ΔABC.

(iv) $\dfrac{R_1^{'}x}{h_a} + \dfrac{R_2^{'}y}{h_b} + \dfrac{R_3^{'}z}{h_c} \geq R$ where ΔABC is not an obtuse triangle and *x, y, z* are the distances from *O* to the 3 sides, respectively.

Proof:

(i) Applying the inequality (M,N) (ii) we obtain $aOB.OC + bOC.OA + cOA.OB \geq abc$ or $R^2 \geq \dfrac{abc}{a+b+c}$.

(ii) Applying the inequality (M,N) we obtain $aGB.GC + bGC.GA + cGA.GB \geq abc$. Since $aGB.GC = 4R_1S_{GBC} = 4R_1\dfrac{S_{ABC}}{3} = 4R_1\dfrac{abc}{3.4R} = R_1\dfrac{abc}{3R}$, $bGC.GA = R_2\dfrac{abc}{3R}$ and $cGA.GB = R_3\dfrac{abc}{3R}$ therefore $R_1\dfrac{abc}{3R} + R_2\dfrac{abc}{3R} + R_3\dfrac{abc}{3R} \geq abc$ or $R_1 + R_2 + R_3 \geq 3R$.

(iii) Applying the inequality (M,N) we have $aIB.IC + bIC.IA + cIA.IB \geq abc$. Since $aIB.IC = 4r_aS_{IBC} = 2r_ara = 4\dfrac{r_a}{h_a}\dfrac{rabc}{4R} = \dfrac{r_a}{h_a}\dfrac{rabc}{R}$, $bIC.IA = \dfrac{r_b}{h_b}\dfrac{rabc}{R}$ and $cIA.IB = \dfrac{r_c}{h_c}\dfrac{rabc}{R}$ we have $\dfrac{r_a}{h_a}\dfrac{rabc}{R} + \dfrac{r_b}{h_b}\dfrac{rabc}{R} + \dfrac{r_c}{h_c}\dfrac{rabc}{R} \geq abc$ or $\dfrac{r_a}{h_a} + \dfrac{r_b}{h_b} + \dfrac{r_c}{h_c} \geq \dfrac{R}{r}$.

(iv) Applying the inequality (M,N) we have $aOB.OC + bOC.OA + cOA.OB \geq abc$. Since $aOB.OC = 4R_1^{'}S_{OBC} = 2R_1^{'}xa = 4R_1^{'}\dfrac{x}{h_a}\dfrac{abc}{4R} = R_1^{'}\dfrac{x}{h_a}\dfrac{abc}{R}$, $bOC.OA = R_2^{'}\dfrac{y}{h_b}\dfrac{abc}{R}$ and $cOA.OB = R_3^{'}\dfrac{z}{h_c}\dfrac{abc}{R}$ we have $\dfrac{R_1^{'}x}{h_a}\dfrac{abc}{R} + \dfrac{R_2^{'}y}{h_b}\dfrac{abc}{R} + \dfrac{R_3^{'}z}{h_c}\dfrac{abc}{R} \geq abc$ or $\dfrac{R_1^{'}x}{h_a} + \dfrac{R_2^{'}y}{h_b} + \dfrac{R_3^{'}z}{h_c} \geq R$. □

Proposition 2.4. Suppose given a triangle *ABC* with the lengths of sides *a, b, c* respectively and *R* is the radius of circumcircle of ΔABC. Let's *I*, J_a, J_b, J_c are the centers of incircle and escribed circles of ΔABC, respectively. Then, with any point *M*, we have

(i) $$\frac{abcMI}{MA.MB.MC} \le \frac{aAI}{MA} + \frac{bBI}{MB} + \frac{cCI}{MC}$$

(ii) $$\frac{MI\sqrt{a+b+c}}{MA.MB.MC} \le \frac{\sqrt{b+c-a}}{\sqrt{bc}MA} + \frac{\sqrt{c+a-b}}{\sqrt{ca}MB} + \frac{\sqrt{a+b-c}}{\sqrt{ab}MC}$$

(iii) $$\frac{MJ_a + MJ_b + MJ_c}{MA.MB.MC} \le \frac{AJ_a + AJ_b + AJ_c}{bcMA} + \frac{BJ_a + BJ_b + BJ_c}{caMB} + \frac{CJ_a + CJ_b + CJ_c}{abMC}$$

(iv) $$\frac{MJ_a.MJ_b + MJ_b.MJ_c + MJ_c.MJ_a}{MA.MB.MC} \le \frac{AJ_a.AJ_b + AJ_b.AJ_c + AJ_c.AJ_a}{bcMA} + \frac{BJ_a.BJ_b + BJ_b.BJ_c + BJ_c.BJ_a}{caMB} + \frac{CJ_a.CJ_b + CJ_b.CJ_c + CJ_c.CJ_a}{abMC}$$

Proof: (i) Applying the inequality (M,N) we have

$$\frac{MI}{MA.MB.MC} \le \frac{AI}{bcMA} + \frac{BI}{caMB} + \frac{CI}{abMC}$$

(ii) Since $IA^2 = \frac{bc(b+c-a)}{a+b+c}$, $IB^2 = \frac{ca(c+a-b)}{a+b+c}$, $IC^2 = \frac{ab(a+b-c)}{a+b+c}$ therefore

$$\frac{MI\sqrt{a+b+c}}{MA.MB.MC} \le \frac{\sqrt{b+c-a}}{\sqrt{bc}MA} + \frac{\sqrt{c+a-b}}{\sqrt{ca}MB} + \frac{\sqrt{a+b-c}}{\sqrt{ab}MC}.$$

(iii) Applying the inequality (M,N) to $n=3,\ s=1$, we have the three inequalities

$$\frac{MJ_a}{MA.MB.MC} \le \frac{AJ_a}{bcMA} + \frac{BJ_a}{caMB} + \frac{CJ_a}{abMC}$$

$$\frac{MJ_b}{MA.MB.MC} \le \frac{AJ_b}{bcMA} + \frac{BJ_b}{caMB} + \frac{CJ_b}{abMC}$$

$$\frac{MJ_c}{MA.MB.MC} \le \frac{AJ_c}{bcMA} + \frac{BJ_c}{caMB} + \frac{CJ_c}{abMC}.$$

On adding the three inequalities, we find the inequality

$$\frac{MJ_a + MJ_b + MJ_c}{MA.MB.MC} \le \frac{AJ_a + AJ_b + AJ_c}{bcMA} + \frac{BJ_a + BJ_b + BJ_c}{caMB} + \frac{CJ_a + CJ_b + CJ_c}{abMC}.$$

(iv) Applying the inequality (M,N) to $n=3,\ s=1$, we have the three inequalities

$$\frac{MJ_a.MJ_b}{MA.MB.MC} \le \frac{AJ_a.AJ_b}{bcMA} + \frac{BJ_a.BJ_b}{caMB} + \frac{CJ_a.CJ_b}{abMC}$$

$$\frac{MJ_b.MJ_c}{MA.MB.MC} \le \frac{AJ_b.AJ_c}{bcMA} + \frac{BJ_b.BJ_c}{caMB} + \frac{CJ_b.CJ_c}{abMC}$$

$$\frac{MJ_c.MJ_a}{MA.MB.MC} \le \frac{AJ_c.AJ_a}{bcMA} + \frac{BJ_c.BJ_a}{caMB} + \frac{CJ_c.CJ_a}{abMC}.$$

On adding the three inequalities, we find the inequality

$$\frac{MJ_a.MJ_b + MJ_b.MJ_c + MJ_c.MJ_a}{MA.MB.MC} \le \frac{AJ_a.AJ_b + AJ_b.AJ_c + AJ_c.AJ_a}{bcMA} + \frac{BJ_a.BJ_b + BJ_b.BJ_c + BJ_c.BJ_a}{caMB} + \frac{CJ_a.CJ_b + CJ_b.CJ_c + CJ_c.CJ_a}{abMC}.$$ □

Corollary 2.5: Given a the triangle *ABC* of the lengths of sides *a, b, c* and *R* is the radius of circumcircle of ΔABC. Denote *O, H* the center of circumcircle and the orthocenter of ΔABC. Then, with any point *M*, we have the inequality:

$$\frac{abcMO.MH}{RMA.MB.MC} \le \frac{aAH}{MA} + \frac{bBH}{MB} + \frac{cCH}{MC}$$

if *M* belongs to the circle with the center *O* and the radius *R*, we obtain the inequality

$$\frac{abcMH}{MA.MB.MC} \le \frac{a\sqrt{4R^2 - a^2}}{MA} + \frac{b\sqrt{4R^2 - b^2}}{MB} + \frac{c\sqrt{4R^2 - c^2}}{MC}.$$

Proof: Applying the inequality (M, N) to $n = 3,\ s = 2$, we have the inequality:

$$\frac{MO.MH}{MA.MB.MC} \le \frac{AO.AH}{bcMA} + \frac{BO.BH}{caMB} + \frac{CO.CH}{abMC}.$$

Thus, we obtain the inequality $\frac{abcMO.MH}{RMA.MB.MC} \le \frac{aAH}{MA} + \frac{bBH}{MB} + \frac{cCH}{MC}$. Since $AH = \sqrt{4R^2 - a^2}$, $BH = \sqrt{4R^2 - b^2}$ and $CH = \sqrt{4R^2 - c^2}$ we obtain $\frac{abcMH}{MA.MB.MC} \le$ $\le \frac{a\sqrt{4R^2 - a^2}}{MA} + \frac{b\sqrt{4R^2 - b^2}}{MB} + \frac{c\sqrt{4R^2 - c^2}}{MC}$ □

Corollary 2.6: Suppose given a triangle *ABC* with the lengths of sides *a, b, c*, respectively. Let *I, G, H* be the center of incircle, the centroid and the orthocenter of ΔABC. Then, with any point *M*, we have the inequality

(i) $$\frac{abcMI^2}{MA.MB.MC} \le \frac{aAI^2}{MA} + \frac{bBI^2}{MB} + \frac{cCI^2}{MC}$$

(ii) $$\frac{abcMG^2}{MA.MB.MC} \le \frac{aAG^2}{MA} + \frac{bBG^2}{MB} + \frac{cCG^2}{MC}$$

(iii) $$\frac{abcMH^2}{MA.MB.MC} \le \frac{a\left(4R^2-a^2\right)}{MA} + \frac{b\left(4R^2-b^2\right)}{MB} + \frac{c\left(4R^2-c^2\right)}{MC}$$

Proof: Applying the inequality (M,N) to $n=3,\ s=2$, $N_1 \equiv N_2 \equiv N$, we have

$$\frac{MN^2}{MA.MB.MC} \le \frac{AN^2}{bcMA} + \frac{BN^2}{caMB} + \frac{CN^2}{abMC}$$

Therefore, we obtain the inequality $\frac{abcMI^2}{MA.MB.MC} \le \frac{aAI^2}{MA} + \frac{bBI^2}{MB} + \frac{cCI^2}{MC}$ and $\frac{abcMG^2}{MA.MB.MC} \le \frac{aAG^2}{MA} + \frac{bBG^2}{MB} + \frac{cCG^2}{MC}$.

Then we have (i) and (ii). If $N \equiv H$ we have (iii): $\frac{abcMH^2}{MA.MB.MC} \le \frac{a\left(4R^2-a^2\right)}{MA} + \frac{b\left(4R^2-b^2\right)}{MB} + \frac{c\left(4R^2-c^2\right)}{MC}$. □

Example. Suppose given a triangle *ABC* of the lengths of sides *a, b, c* respectively. *R* is the radius of circumcircle; r_1, r_2, r_3 are the radii of escribed circles correspondence to vertices *A, B, C*, respectively. Let d_a, d_b, d_c the distances from the center of circumcircle to the center of escribed circles. Then, with any point *D* belong to the circumcircle of ΔABC we have the inequality:

(i) $$\frac{\frac{d_a d_b d_c}{\sqrt{a+b+c}}}{R^3} \le \frac{\sqrt{bc}}{x\sqrt{b+c-a}} + \frac{\sqrt{ca}}{y\sqrt{c+a-b}} + \frac{\sqrt{ab}}{z\sqrt{a+b-c}} + \frac{DJ_a.DJ_b.DJ_c}{xyz\sqrt{a+b+c}}$$

(ii) $$\sqrt{\frac{(R+2r_1)(R+2r_2)(R+2r_3)}{R^3(a+b+c)}} \le \frac{\sqrt{bc}}{\sqrt{b+c-a}} + \frac{\sqrt{ca}}{\sqrt{c+a-b}} + \frac{\sqrt{ab}}{\sqrt{a+b-c}} + \frac{DJ_a.DJ_b.DJ_c}{xyz\sqrt{a+b+c}}.$$

Proof: (i) We consider $M \equiv O$.

Since $$\begin{cases} J_aA^2 = \frac{bc(a+b+c)}{b+c-a},\ J_aB^2 = \frac{ca(a+b-c)}{b+c-a}, J_aC^2 = \frac{ab(a-b+c)}{b+c-a} \\ J_bA^2 = \frac{bc(b+a-c)}{c+a-b},\ J_bB^2 = \frac{ca(a+b+c)}{c+a-b}, J_bC^2 = \frac{ab(b+c-a)}{c+a-b} \\ J_cA^2 = \frac{bc(c+a-b)}{a+b-c},\ J_cB^2 = \frac{ca(c+b-a)}{a+b-c}, J_cC^2 = \frac{ab(a+b+c)}{a+b-c} \end{cases}$$ therefore

we obtain $$\frac{\frac{d_a d_b d_c}{\sqrt{a+b+c}}}{R^3} \le \frac{\sqrt{bc}}{x\sqrt{b+c-a}} + \frac{\sqrt{ca}}{y\sqrt{c+a-b}} + \frac{\sqrt{ab}}{z\sqrt{a+b-c}} + \frac{DJ_a.DJ_b.DJ_c}{xyz\sqrt{a+b+c}}.$$

(ii) Since $d_a^2 = R^2 + 2Rr_1$, $d_b^2 = R^2 + 2Rr_2$, $d_c^2 = R^2 + 2Rr_3$ therefore

$$\sqrt{\frac{(R+2r_1)(R+2r_2)(R+2r_3)}{R^3(a+b+c)}} \le \frac{\sqrt{bc}}{\sqrt{b+c-a}} + \frac{\sqrt{ca}}{\sqrt{c+a-b}} + \frac{\sqrt{ab}}{\sqrt{a+b-c}} + \frac{DJ_a.DJ_b.DJ_c}{xyz\sqrt{a+b+c}}. \quad \square$$

Proposition 2.7. Let $A_1A_2...A_n$ be a polygon inscribed in the circle with the center *O* and radius *R*. Then, with any $s < n$ points $N_1...N_s$ in the plane $A_1A_2...A_n$, we have the inequality

$$\sum_{k=1}^{n} \frac{\prod_{i=1}^{s} A_kN_i}{\prod_{i=1,i\neq k}^{n} A_kA_i} \ge \frac{\prod_{i=1}^{s} ON_i}{R^{n-1}}.$$

If $R = 1$ we obtain $\sum_{k=1}^{n} \frac{\prod_{i=1}^{s} A_kN_i}{\prod_{i=1,i\neq k}^{n} A_kA_i} \ge \prod_{i=1}^{s} ON_i$.

If $n = 3$, $s = 1$ and $a_1 = A_2A_3$, $a_2 = A_3A_1$, $a_3 = A_1A_2$ we obtain the inequality

$$a_1A_1N + a_2A_2N + a_3A_3N \ge 4S_{A_1A_2A_3}\frac{ON}{R}.$$

Proof: Applying the inequality (M,N) with $M \equiv O$, we have the inequality

$$\sum_{k=1}^{n} \frac{\prod_{i=1}^{s} A_kN_i}{\prod_{i=1,i\neq k}^{n} A_kA_i} \ge \frac{\prod_{i=1}^{s} ON_i}{R^{n-1}}$$

Let $R = 1$ we obtain $\sum_{k=1}^{n} \frac{\prod_{i=1}^{s} A_kN_i}{\prod_{i=1,i\neq k}^{n} A_kA_i} \ge \prod_{i=1}^{s} ON_i$. □

Now, we illustrate the advantage of the identity (M,N) by addressing several important problems of elementary Geometry. Firstly, we use the functions *sin* and *cosin* to create the identity under the form of trigonometry.

Without generality, we can assume that the radius *R* of the circle *C* equal to 1. Suppose that every point A_k has affixe $a_k = \cos\alpha_k + i\sin\alpha_k$, and *M* has affixe $z = \cos u + i\sin u$ and every N_h has affixe $z_h = \cos u_h + i\sin u_h$. From Lagrange interpolation formula, we have

$$\frac{\prod_{j=1}^{s}(z-z_j)}{\prod_{t=1}^{n}(z-a_j)} = \sum_{k=1}^{n} \frac{\prod_{j=1}^{s}(a_k-z_j)}{(z-a_k)\prod_{t\neq k}(a_k-a_t)}$$

or

$$\frac{\prod_{j=1}^{s} 2i\sin\frac{u-u_j}{2} e^{\frac{i(u+u_j)}{2}}}{\prod_{t=1}^{n} 2i\sin\frac{u-\alpha_t}{2} e^{\frac{i(u+\alpha_t)}{2}}} = \sum_{k=1}^{n} \frac{\prod_{j=1}^{s} 2i\sin\frac{\alpha_k-u_j}{2} e^{\frac{i(\alpha_k+u_j)}{2}}}{2i\sin\frac{u-\alpha_k}{2} e^{\frac{i(u+\alpha_k)}{2}} \prod_{t\neq k} 2i\sin\frac{\alpha_k-\alpha_t}{2} e^{\frac{i(\alpha_k+\alpha_t)}{2}}}$$

We reduce all the factors $2i$, $e^{\frac{iu_j}{2}}$ and $e^{\frac{i\alpha_t}{2}}$, obtain the relation

$$\frac{\prod_{j=1}^{s} \sin\frac{u-u_j}{2}}{\prod_{t=1}^{n} \sin\frac{u-\alpha_t}{2}} = \sum_{k=1}^{n} \frac{\prod_{j=1}^{s} \sin\frac{\alpha_k-u_j}{2}}{\sin\frac{u-\alpha_k}{2} \prod_{t\neq k} \sin\frac{\alpha_k-\alpha_t}{2}} e^{\frac{i(s+1-n)(\alpha_k-u)}{2}}$$

From this relation, we deduce two identities below:

$$\frac{\prod_{j=1}^{s} \sin\frac{u-u_j}{2}}{\prod_{t=1}^{n} \sin\frac{u-\alpha_t}{2}} = \sum_{k=1}^{n} \frac{\prod_{j=1}^{s} \sin\frac{\alpha_k-u_j}{2}}{\sin\frac{u-\alpha_k}{2} \prod_{t\neq k} \sin\frac{\alpha_k-\alpha_t}{2}} \cos\frac{(s+1-n)(\alpha_k-u)}{2}$$

and

$$\sum_{k=1}^{n} \frac{\prod_{j=1}^{s} \sin\frac{\alpha_k-u_j}{2}}{\sin\frac{u-\alpha_k}{2} \prod_{t\neq k} \sin\frac{\alpha_k-\alpha_t}{2}} \sin\frac{(s+1-n)(\alpha_k-u)}{2} = 0$$

From this result, we build the identities under the form of trigonometry and geometry for the inequality (M,N) as following:

Proposition 2.8. Assume that the polygon $A_1A_2...A_n$ is inscribed in the circle with the center *O* and radius *R*. Taking $s+1$ points $N_1...N_s$ and *M* also belonging to this circle *C*. Assuming that the coordinate $A_k(\cos\alpha_k;\sin\alpha_k)$, $k=1,2,...,n$; the coordinate $N_j(\cos u_j;\sin u_j)$, $j=1,2,...,s$ and the coordinate $M(\cos u;\sin u)$. Then, we will have these identities

(i) $$\frac{\prod_{j=1}^{s} \sin\frac{u-u_j}{2}}{\prod_{t=1}^{n} \sin\frac{u-\alpha_t}{2}} = \sum_{k=1}^{n} \frac{\prod_{j=1}^{s} \sin\frac{\alpha_k-u_j}{2}}{\sin\frac{u-\alpha_k}{2} \prod_{t\neq k} \sin\frac{\alpha_k-\alpha_t}{2}} \cos\frac{(s+1-n)(\alpha_k-u)}{2}$$

(ii) $$\sum_{k=1}^{n}\frac{\prod_{j=1}^{s}\sin\frac{\alpha_k-u_j}{2}}{\sin\frac{u-\alpha_k}{2}\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}}\sin\frac{(s+1-n)(\alpha_k-u)}{2}=0$$

(iii) $$\sum_{k=1}^{3}\frac{\sin\frac{\alpha_k-u_1}{2}}{\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}}=0 \text{ if } n=3,\ s=1$$

(iv) $$\frac{\prod_{j=1}^{n-1}\sin\frac{u-u_j}{2}}{\prod_{t=1}^{n}\sin\frac{u-\alpha_t}{2}}=\sum_{k=1}^{n}\frac{\prod_{j=1}^{n-1}\sin\frac{\alpha_k-u_j}{2}}{\sin\frac{u-\alpha_k}{2}\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}} \text{ if } s=n-1$$

(v) $$\frac{\prod_{j=1}^{n-2}\sin\frac{u_j}{2}}{\prod_{t=1}^{n}\sin\frac{\alpha_t}{2}}=\sum_{k=1}^{n}\frac{\prod_{j=1}^{n-2}\sin\frac{\alpha_k-u_j}{2}}{\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}}\cot\frac{\alpha_k}{2} \text{ and } \sum_{k=1}^{n}\frac{\prod_{j=1}^{n-2}\sin\frac{\alpha_k-u_j}{2}}{\prod_{t\neq k}\sin\frac{\alpha_k-\alpha_t}{2}}=0 \text{ if } s=n-2 \text{ and}$$
$u=0$.

Remark 2.9. If the quadrilateral *ABCD* is inscribed in the circle we have

$$\frac{DA}{bc}+\frac{DC}{ab}+\frac{DB}{ca}$$

by (iii) or $\frac{aDA^2}{DA}+\frac{cDC^2}{DC}=\frac{bDB^2}{DB}$. Moreover, we have

$$DA^2.DB.DC.a+DC^2.DA.DB.c=DB^2.DC.DA.b$$

Hence $DA^2S_{DBC}+DC^2S_{DAB}=DB^2D_{DCA}$ [**Feuerbach**].

Proposition 2.10. Suppose the polygon $A_1A_2...A_n$ is inscribed in the circle with the radius $R=1$. Taking $s+1$ points $N_1...N_s$ and M also belonging to this circle *C*. Assuming that the coordinate $A_k(\cos\alpha_k;\sin\alpha_k)$, $k=1,2,...,n$; the coordinate $N_j(\cos u_j;\sin u_j)$, $j=1,2,...,s$ and the coordinate $M(\cos u;\sin u)$. Then, with the proper choices of + or - we will have the identities

(i) $$\frac{\prod_{j=1}^{s}MN_j}{\prod_{t=1}^{n}MA_t}=\sum_{k=1}^{n}\frac{\pm\prod_{j=1}^{s}A_kN_j}{MA_k\prod_{t\neq k}A_kA_t}\cos\frac{(s+1-n)(\alpha_k-u)}{2},\ (M,N)$$

(ii) $$\sum_{k=1}^{n} \frac{\pm\prod_{j=1}^{s} A_kN_j}{MA_k \prod_{t\neq k} A_kA_t} \sin\frac{(s+1-n)(\alpha_k - u)}{2} = 0$$

(iii) $$\frac{\prod_{j=1}^{n-2} MN_j}{\prod_{t=1}^{n} MA_t} = \sum_{k=1}^{n} \frac{\pm\prod_{j=1}^{n-2} A_kN_j}{\prod_{t\neq k} A_kA_t} \cot\frac{\alpha_k}{2} \text{ and } \sum_{k=1}^{n} \frac{\pm\prod_{j=1}^{n-2} A_kN_j}{\prod_{t\neq k} A_kA_t} = 0.$$

Corollary 2.11. Assume that the points $A_1A_2...A_n$, *M* in order belong to the circle *C* with the center *O*. Then, we have the identities

(i) $$\sum_{r=1}^{n} (-1)^r \frac{\cos(n-1)\angle MA_{r+1}A_r}{MA_r \prod_{k\neq r} A_rA_k} = \frac{1}{\prod_{k=1}^{n} MA_k}$$

(ii) $$\sum_{r=1}^{n} (-1)^r \frac{\sin(n-1)\angle MA_{r+1}A_r}{MA_r \prod_{k\neq r} A_rA_k} = 0$$

Proof: These identities follow from the identity (M,N) with $s=0$.

Corollary 2.12. Let the quadrilateral *ABCD* be inscribed in the circle *C* with the center *O*. Let *a* = *BC*, *b* = *CA*, *c* = *AB*. Then, we have two identities:

(i) $$\frac{a\cos(OD,OA)}{DA} - \frac{b\cos(OD,OB)}{DB} + \frac{c\cos(OD,OC)}{DC} = -\frac{abc}{DA.DB.DC}$$

(ii) $$\frac{a\sin(OD,OA)}{DA} + \frac{c\sin(OD,OC)}{DC} = \frac{b\sin(OD,OB)}{DB}$$

Proof: These identities follow from the identity (M,N) if $n=3,\ s=0$.

3. CONJECTURE

Despite of not having been proven yet, these following results are still hoped to be true:

Open Problem 3.1. Suppose given a triangle *ABC* with the lengths of sides *a, b, c* respectively and *R* is the radius of circumcircle of ΔABC. Let's J_a, J_b, J_c are the centers of escribed circles of ΔABC, respectively. Then, with any point *M*, we have

(i) $$\frac{MJ_a.MJ_b.MJ_c}{MA.MB.MC} \leq \frac{AJ_a.AJ_b.AJ_c}{bcMA} + \frac{BJ_a.BJ_b.BJ_c}{caMB} + \frac{CJ_a.CJ_b.CJ_c}{abMC}$$

(ii) $$\frac{\dfrac{MJ_a.MJ_b.MJ_c}{\sqrt{a+b+c}}}{MA.MB.MC} \le \frac{\dfrac{\sqrt{bc}}{\sqrt{b+c-a}}}{MA} + \frac{\dfrac{\sqrt{ca}}{\sqrt{c+a-b}}}{MB} + \frac{\dfrac{\sqrt{ab}}{\sqrt{a+b-c}}}{MC}$$

Open Problem 3.2. Giving a triangle *ABC* with the lengths of sides *a, b, c* and *R* is the radius of circumscribed circle; r_1, r_2, r_3 are the radii of escribed circles. Let's d_a, d_b, d_c the distances from the center of circumscribed circle to the center of escribed circles. Then we always have the inequality:

(i) $$\frac{\dfrac{d_a d_b d_c}{\sqrt{a+b+c}}}{R^3} \le \frac{\sqrt{bc}}{x\sqrt{b+c-a}} + \frac{\sqrt{ca}}{y\sqrt{c+a-b}} + \frac{\sqrt{ab}}{z\sqrt{a+b-c}}$$

(ii) $$\sqrt{\frac{(R+2r_1)(R+2r_2)(R+2r_3)}{R^3(a+b+c)}} \le \frac{\sqrt{bc}}{\sqrt{b+c-a}} + \frac{\sqrt{ca}}{\sqrt{c+a-b}} + \frac{\sqrt{ab}}{\sqrt{a+b-c}}$$

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