ORIGINAL PAPER

THE TOPOLOGY OF THE SPACES OF MELLIN WHITTAKER TRANSFORMABLE DISTRIBUTIONS

R.V. KENE [1], ALKA S. GUDADHE[1]

Manuscript received: 17.10.2011; Accepted paper: 06.11.2011;

Published online: 01.12.2011.

Abstract. *In this paper we have introduced Gelfand Shilove type spaces of Mellin Whittaker transformable distributions. We have also discussed their interrelation and some topological properties of these spaces.*

Keywords: *topology, spaces, transformable distribution.*

1. INTRODUCTION

Zemanian [4] had developed the theory of generalized integral transform originally found in the work of Schwartz. His work is concerned mostly with the space D.ε.S. and their duals. Gelfand, Shilov [2] had investigated new type of fundamental spaces called $S_{\propto}$ and S^{β} spaces by imposing conditions not only on the decrease of the fundamental functions at infinity but also on the growth of their derivatives as the order of the derivative increases. He also used them to solve Cauchy problem.

In other study it was defined distributional generalized Mellin-Whittaker transform of the functions in the space MW', which is the dual space of MW. Motivated from the work of Gelfand-Shilov, we wish to extend the generalized Mellin-Whittaker transform, given in the same paper, to the spaces $M_{\propto}W'$ and $M^{\beta}W'$, for which, in this paper we have studied the spaces of the type $M_{\propto}W$ and $M^{\beta}W$ and their interrelationship with topological properties.

2. DIFFERENT TYPES OF TESTING FUNCTION SPACES

2.1. TESTING FUNCTION SPACE $M_{\propto}W$ AND $M^{\beta}W$:

The space $M_{\propto}W$, $\propto \geq 0$ consist of all infinitely differentiable functions $\phi(x,t)$, where $0 < x < \infty$, $0 < t < \infty$ satisfying the inequality,

$$\gamma_{k,l}\,\phi(x,t) = \sup_{\substack{0<x<\infty \\ 0<t<\infty}} \left|\lambda_{a,b}(x,t)x^{k+1}D_x^k(-tD_t)^l\,(t\,\phi(x,t))\right| \leq C_k A^k k^{k\alpha} \quad (k = 0,1,2 \ldots\ldots\ldots\ldots)$$

Where the constants A and C_k depends on the function $\phi(x,t)$.

Here the space $M_{\propto}W$ and $M^{\beta}W$ are similar.

[1] Government Vidarbha Institute of Science and Humanties, Department of Mathematics, 444701, Amravati, India. E-mail: alka.gudadhe@gmail.com.

2.2. TESTING FUNCTION SPACES $MW_{\propto}$ AND MW^{B}:

An infinitely differentiable function $\phi(x,t)$, where $0 < x < \infty$, $0 < t < \infty$ is said to belong to the space $MW_{\propto}$, if

$$\sigma_{k,l}\,\phi(x,t) = \sup_{\substack{0<x<\infty \\ 0<t<\infty}} |\lambda_{a,b}(x,t)x^{k+1}D_x^k(-tD_t)^l(t\,\phi(x,t))|$$
$$\leq C_l A^l l^{l\alpha} \qquad (l = 0,1,2\ldots\ldots\ldots).$$

Where the constants A and C_l depended on the function $\phi\,(x,t)$.

The testing function spaces $MW_{\propto}$ and MW^{β} are also similar.

2.3. TESTING FUNCTION SPACES $M_{\propto,M}W$ AND $M^{\beta,m}w$:

The space $M_{\propto,m}\,W$ consists of those functions $\phi\,(x,t)$, which satisfy the inequality,

$$\xi_{k,l}\,\phi(x,t) = \sup_{\substack{0<x<\infty \\ 0<t<\infty}} |\lambda_{a,b}(x,t)x^{k+1}D_x^k(-tD_t)^l(t\,\phi(x,t))|$$
$$\leq C_k(A+m)^k k^{k\alpha} \quad (k = 0,1,2\ldots\ldots\ldots\ldots).$$

Since space $M_{\propto}W$ and $M^{\beta}W$ are similar, so that spaces $M_{\propto,m}\,W$ and $M^{\beta,m}W$ are also similar.

2.4. TESTING FUNCTION SPACES $MW_{\propto,N}$ AND $MW^{\beta,n}$:

The space $MW_{\propto,n}$ consists of those functions $\phi\,(x,t)$. which satisfy the inequality.

$$\eta_{k,l}\,\phi(x,t) = \sup_{\substack{0<x<\infty \\ 0<t<\infty}} |\lambda_{a,b}(x,t)x^{k+1}D_x^k(-tD_t)^l(t\,\phi(x,t))|$$
$$\leq C_l(B+n)^l l^{l\alpha} \quad (l = 0,1,2\ldots\ldots\ldots\ldots\ldots)$$

Because spaces $MW_{\propto}$ and MW^{β} are similar, so that spaces $MW_{\propto,n}$ and $MW^{\beta,n}$ are also similar.

The spaces defined in (2.1-2.4) are equipped with their natural Hausdrof locally convex topologies, denoted by $T_{\propto}$ and T^{β}, $T_{\propto,m}$ and $T^{\beta,n}$ respectively. These topologies are generated by the total families of seminorms $\{r_{k,l}\}$, $\{\sigma_{k,l}\}$, $\{\xi_{k,l}\}$ and $\{\eta_{k,l}\}$ respectively.

3. PROPERTIES OF S_{α} TYPE SPACES

In this section we have given some results on the topological structure of the spaces defined in the previous section.

3.1 PROPOSITION:

If $\propto_1 < \propto_2$ *then* $M\alpha_1 W \subset M\alpha_2 W$

Proof : Let $\phi \in M\alpha_1 W$

$$\gamma_{k,l}\,\phi(x,t) = \sup_{\substack{0<x<\infty \\ 0<t<\infty}} |\lambda_{a,b}(x,t)x^{k+1}D_x^k(-tD_t)^l(t\,\phi(x,t))|$$
$$\leq C^k A^k k^{k\alpha_1}$$

Since, $\alpha_1 < \alpha_2$

$$\leq C^k A^k k^{k\alpha_2}$$

Hence $\phi\,(x,t) \in M_{\alpha_2}$

Consequently $M_{\alpha_1} W \subset M_{\alpha_2} W$.

3.2. PROPOSITION:

If $m_1 < m_2$ then $M_{\alpha,m_1} W \subset M_{\alpha,m_2} W$

Proof : For $\phi \in M_{\alpha,m_1} W$

$$\xi_{k,l}\, \phi(x,t) = \sup_{\substack{0<x<\infty \\ 0<t<\infty}} |\lambda_{a,b}(x,t) x^{k+1} D_x^k (-tD_t)^l (t\, \phi(x,t))|$$

$$\le C_k (A+m_1)^k k^{k\alpha}$$

Since $m_1 < m_2$ we have,

$$\le C_k (A+m_2)^k k^{k\alpha} .$$

Thus, $\phi(x,t) \in M_{\alpha,m_2} W$

So, $M_{\alpha,m_1} W \subset M_{\alpha,m_2} W$

Hence the proof.

Now we pay attention to the inductive limit space in the next proposition.

3.3. PROPOSITION:

$MW = \bigcup_{\alpha_i=1}^{\infty} M_{\alpha_i} W$ and if the space MW is equipped with the strict inductive limit topology defined by injection map from MW_{α_i} to MW then the sequence { ϕ_n} in MW converges to zero if and only if { ϕ_n} is contained in some MW_{α_i} and converges there-in to zero.

Proof: The above proposition is immediate consequence of theorem given in [3] . Therefore if we prove $MW = \bigcup_{\alpha_i=1}^{\infty} M_{\alpha_i} W$ then Seq. { ϕ_n} in MW converges to zero if { ϕ_n} is contained in some MW_{α_i} and converges there.

Clearly

$$MW = \bigcup_{\alpha_i=1}^{\infty} M_{\alpha_i} W \subset MW$$

For proving other inclusion, Let $\phi \in MW$ then

$$\gamma_{k,l}\, \phi(x,t) = \sup_{\substack{0<x<\infty \\ 0<t<\infty}} |\lambda_{a,b}(x,t) x^{k+1} D_x^k (-tD_t)^l (t\, \phi(x,t))|$$

Is bounded by some number. We can choose the integer α_m such that

$$\gamma_{k,l}\ \phi(x,t) \le C_k A^k k^{k\alpha_m}$$

$\therefore \quad \phi(x,t) \in M_{\alpha,m} W$, for some integer α_m .

Hence $MW \subset \bigcup_{\alpha_i=1}^{\infty} M_{\alpha i} W$

Thus $MW = \bigcup_{\alpha_i=1}^{\infty} M_{\alpha i} W$.

4. THEOREMS ON S_α TYPE SPACES

To justify the study of these spaces, we prove the following theorem.

4.1. THEOREM:

The space *D*(*I*) is a subspace of $M_\alpha W$.

Proof: Let $\phi(x,t) \in D(I)$

Consider, $C_k = \sup_{\substack{o<x<\infty \\ o<t<\infty}} \left| \lambda_{a,b}(x,t) x D_x^k \left(-tD_t)^l\right) \left(t\phi(x,t)\right) \right|$

$$L = \sup\left\{ x : (x,t) \in \sup \phi \right\}$$

$$\gamma_{k,l}\ \phi(x,t) = \sup_{\substack{o<x<\infty \\ o<t<\infty}} \left| \lambda_{a,b}(x,t) x^{k+1} D_x^k \left(-tD_t)^l\right) \left(t\phi(x,t)\right) \right|$$

$$= \sup_{\substack{o<x<\infty \\ o<t<\infty}} \left| \lambda_{a,b}(x,t) x D_x^k x^k \left(-tD_t)^l\right) \left(t\phi(x,t)\right) \right|$$

$$\le C_k . L^k$$

$$= C_k . L^k \frac{k^{k\alpha}}{k^{k\alpha}} \frac{A^k}{A^k}$$

$$= C_k \left(\frac{L}{Ak^\alpha} \right)^k A^k k^{k\alpha} \tag{4.1}$$

Since $\left(\frac{L}{Ak^\alpha} \right) \le 1$ if and only if $k \ge \left(\frac{L}{A} \right)^{1/\alpha}$.

Define $k_o = \left[\left(\frac{L}{A} \right)^{1/\alpha} \right] + 1$, therefore for $k > k_o$, we have

$$\gamma_{k,l}\ \phi(x,t) \le C_k A^k k^{k\alpha} \tag{4.2}$$

If $k \le k_o$ Let us write,

$$C = \max\left[\frac{L}{A}, \left(\frac{L}{A_2^\alpha} \right)^2 ----- \left(\frac{L}{A_{k_o}^\alpha} \right)^{k_o} \right].$$

Then again from (4.1)

$$\gamma_{k,l}\ \phi(x,t) \le C C_k A^k k^{k\alpha} \tag{4.3}$$

Hence, the inequalities (4.2), (4.3) together yield,

$$\gamma_{k,l}\ \phi(x,t) \le C_k A^k k^{k\alpha} \quad \forall k \ge 0$$

Hence, $\phi \in M_\alpha W$

Consequently $D(I) \subset M_\alpha W$.

4.2. THEOREM:

MW_α is a Frechet space.

Proof: As the family $D_{a,b}$ of seminorms $\{\sigma_{a,b,l}^k\}_{k,l=0}^\infty$ generating $T_{a,b}$ is countable, it suffices to prove the completeness of the space MW_α .

Let $\{\phi_\nu\}_{\nu=1}^\infty$ is a Cauchy sequence in MW_α .

Hence for a given $\varepsilon > 0$ there exists an $N_{k,l} = N$ such that for $\nu, \mu \geq N_{k,l} = N$

$$\sigma_{k,l}^k \left(\phi_\nu - \phi_\mu\right) = \sup_{\substack{o<x<\infty \\ o<t<\infty}} \left| \lambda_{a,b}(x,t) x^{k+1} D_x^k \left(-tD_t)^l\right) \left[t\left(\phi_\nu - \phi_\mu\right)\right] \right| < \varepsilon \tag{4.4}$$

In particular for $k = l = 0$ for $\nu, \mu \geq N$

$$= \sup_{\substack{o<x<\infty \\ o<t<\infty}} \left| \lambda_{a,b}(x,t) x.t \left[\phi_\nu(x,t) - \phi_\mu(x,t)\right] \right| < \varepsilon . \tag{4.5}$$

Consequently, for fixed (x,t) in *I*, $[\phi_\nu(x,t)]$ is a Cauchy sequence.

Let $\phi(x,t)$ be the limit function of $\{\phi_\nu(x,t)\}$.

Using (4.5), we can easily deduce that $\{\phi_\nu\}$ converges to ϕ uniformly on *I*.

Thus ϕ is continuous.

Moreover, repeated use of (4.4) for different values of *k*, *l* and the use of the above proposition yield that ϕ is smooth i.e. $\phi \in E_+$.

Further from (4.4) we get,

$$\sigma_{a,b,l}^k(\phi_\nu) \leq \sigma_{a,b,l}^k(\phi_N) + \varepsilon \qquad \forall \nu \geq N$$
$$\leq C_l A^l l^{l\alpha} + \varepsilon .$$

Let $\nu_k \to \infty$ and ε is arbitrary we get,

$$\sigma_{a,b,l}^k(\phi) = \sup_{\substack{o<x<\infty \\ o<t<\infty}} \left| \lambda_{a,b}(x,t) x^{k+1} D_x^k (-tDt)^l (t\phi(x,t)) \right|$$
$$\leq C_l A^l l^{l\alpha}$$

Hence $\phi \in MW_\alpha$.

This proves the completeness of MW_α and our theorem is proved.

4.3. THEOREM:

Prove that $MW_{\alpha} = \bigcup_{m=1}^{\infty} MW_{\alpha,m}$

Proof: - $MW_{a,b,\alpha} = \bigcup_{m=1}^{\infty} MW_{a,b,\alpha,m}$

Clearly $\bigcup_{m=1}^{\infty} MW_{a,b,\alpha,m} \subset MW_{a,b,\alpha}$

For proving the other inclusion, let $\phi \in MW_{a,b,\alpha}$, then

$$\sigma_{a,b,k}^{l}(\phi) = \sup_{\substack{o<x<\infty \\ o<t<\infty}} \left| \lambda_{a,b}(x,t) x^{k+1} D_x^k (-tDt)^l (t\phi(x,t)) \right|$$

$$\leq C_l B^l l^{l\alpha} \tag{4.6}$$

where B is some constant.

Choose an integer $m = m_B$ and $\delta > 0$

Such that $C_l B^l \leq C_l (m+\delta)^l$

Then from (4.6) we immediately get $\phi \in MW_{\alpha,m}$

Implying that

$$MW_{\alpha} \subset \bigcup_{m=1}^{\infty} MW_{\alpha,m}$$

Hence the theorem.

REFERENCES:

[1] Dieudonne, J., *History of functional analysis*, North Holland Amsterdam, NY, Oxford, 1987.

[2] Gelfand, I.M., Shilov, G.E., *Generalized Functions*, II, Academic Press, New York, 1967.

[3] Gudadhe, A.S., Ph.D. Thesis, Nagpur University, 1992.

[4] Zemanian, A.H., *Generalized Integral Transformation*, Interscience Publishers, New York, 1968.