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BINARY OPERATIONS ASSOCIABLE WITH A GROUP OPERATION

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Abstract. *Conditions under which two binary operations on a set M are associable* $(x\circ(y*z)=(x\circ y)*z,\ \ z,y,z\in M)$ *were given by J. Dhombres in the case when the operations are associative. We characterize the associability of two group operations by using the binary reduces (in Hosszú sense) of an n-group. We show that by the juxtaposition of two such operations, an operation of 3-group is obtained.*

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1. INTRODUCTION

Using functional equations J. Dhombres determines in [2] the binary operations which are associative, associable and commutable with a quasigroup or monoid operation.

We will approach this problem from another point of view and remark a natural relation of this problem with the two reduced *n*-groups (in Hosszú sense) of an *n*-group.

We recall some notations and results which are used in the paper.

1.1. (See [1]) If (G,φ) is a group, $\alpha:G\to G$ an automorphism, $a\in G$ a fixed element such that $\alpha^{n-1}(x)=a\circ x\circ a^{-1}$, $x\in G$ is an inner automorphism and $\alpha(a)=a$, then the *n*-ary operation $\varphi:G^n\to G$ defined by

$$\varphi(x_1,x_2,...,x_n)=x_1\circ\alpha(x_2)\circ...\circ\alpha^{n-1}(x_n)\circ a,\ x_1,x_2,...,x_n\in G$$

determines on *G* an *n*-group structure, which is called an *n*-extension of the group $(G,\circ)$ and is denoted by

$$(G,\varphi)=Ext_{\alpha,a}(G,\circ).$$

1.2. (See [3]) If (G,φ) is an *n*-group then for all $u\in G$, the binary operation $*:G\times G\to G$ defined by:

$$x*y=\varphi(x,\underset{n-3}{u},\bar{u},y),\ x,y\in G$$

determines on *G* a group structure $(G,*)=\mathrm{Re}\,d_u(G,\varphi)$ which is called a reduced group in Hosszú sense. The element $\bar{u}$ is called the skew element of *u* and is defined by the relations

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$$\varphi(x, \underset{n-2}{u}, \bar{u}) = \varphi(\underset{n-2}{u}, \bar{u}, x) = x \text{ for all } x \in G.$$

1.3. (See [3]) If (G,φ) is an n-group and $(G,*) = Red\ (G,\varphi)$ is a reduced group, then the function $\alpha_u : G \to G$,

$$\alpha_u(x) = \varphi(u, x, \underset{n-3}{u}, \bar{u}),\ x \in G$$

is an automorphism, $\alpha_u(a) = a$, when $a = \varphi(\underset{n}{u})$ and $\alpha_u^{n-1}(x) = a * x * a^{-1}$, $x \in G$. Thus the pair (α_u, a) is a pair of extension and $(G,\varphi) = Ext_{\alpha_u, a}(G,*)$.

2. MAIN RESULTS

Let G be a set, $\circ : G \times G \to G$ and $* : G \times G \to G$ two binary operations on G.

Definition 2.1. (See [6]) The binary operations $\circ$ and $*$ are called reciprocally associable if the following relations hold:

a) $x * (y \circ z) = (x * y) \circ z$

b) $x \circ (y * z) = (x \circ y) * z$

for all $x, y, z \in G$

In the case when $(G,\circ)$ is a group, our goal is to characterize the binary operations $*$ on G, which are associable with the given operation $\circ$.

Theorem 2.2. *If $(G,\circ)$ is a group and the binary operation $* : G \times G \to G$ is reciprocally associable with the operation $\circ$ then $(G,*)$ is a group, isomorphic to the group $(G,\circ)$.*

Proof: If $u \in G$ is the unit element in $(G,\circ)$ we show that

$$x * y = x \circ (u * u) \circ y, \text{ for all } x, y \in G. \tag{1}$$

We have

$$\begin{aligned} x \circ (u * u) \circ y &= x \circ ((u * u) \circ y) = x \circ (u * (u \circ y)) \\ &= x \circ (u * y) = (x \circ u) * y = x * y. \end{aligned}$$

Associativity:

$$\begin{aligned} x * (y * z) &\overset{(1)}{=} x \circ (u * u) \circ (y \circ (u * u) \circ z) \\ &= (x \circ (u * u) \circ y) \circ (u * u) \circ z = (x * y) * z, \text{ for all } x, y, z \in G. \end{aligned}$$

Unit element: If in (1) we take $y = u$, we obtain successively

$$x * u = x \circ ((u * u) \circ u) = x \circ (u * u);$$

$$(x * u) \circ (u * u)^{-1} = x;$$

$$x * (u \circ (u * u)^{-1}) = x;$$

$$x * (u * u)^{-1} = x,\ x \in G.$$

So $(u * u)^{-1}$ is a right unit element in $(G,*)$.

If in (1) we take $x = u$ we obtain

$$(u * u)^{-1} * y = y,\ y \in G.$$

So $(u * u)^{-1}$ is a left unit element in $(G,*)$. Using the associativity we conclude that $v = (u * u)^{-1}$ is the unit element in $(G,*)$.

Symmetric element: We show that $x' = v \circ x^{-1} \circ v$ is the symmetric of x in $(G,*)$.

We have

$$x * x' = x * (v \circ x^{-1} \circ v) = (x * v) \circ (x^{-1} \circ v) = x \circ (x^{-1} \circ v) = (x \circ x^{-1}) \circ v = v,$$

and similarly $x' * x = v$.

The isomorphism: We prove that the function $f : (G,\circ) \to (G,*)$

$$f(x) = v \circ x,\ \ x \in G$$

is a group isomorphism.

We have

$$f(x \circ y) = v \circ (x \circ y)$$

and

$$f(x) * f(y) = (v \circ x) * (v \circ y) = v \circ (x * (v \circ y)) = v \circ ((x * v) \circ y) = v \circ (x \circ y),$$

so

$$f(x \circ y) = f(x) * f(y),\ x, y \in G.$$

Remark 2.3. (See [4]) From the proof of the previous theorem we conclude:

- $x * y = x \circ (u * u) \circ y = x \circ v^{-1} \circ y,\ x, y \in G$
- $(u * u)^{-1}$ is the unit element in $(G,*)$
- $x' = v \circ x^{-1} \circ v,\ x \in G$ where u is the unit element in $(G,\circ)$, v is the unit element in $(G,*)$, x^{-1} is the symmetric of x in $(G,\circ)$ and x' is the symmetric of x in $(G,*)$.

In order to realize the relation with the *n*-groups we remind the definition (see [5]).

Definition 2.4. The groups $(G,\circ)$ and $(G,*)$ are called simultaneously *n*-extensible if there exists an *n*-group (G,φ) and two elements $u, v \in G$ such that these groups are the Hosszú reduced groups

$$(G,\circ) = Red_{\,u}(G,\varphi) \text{ and } (G,*) = \mathrm{Red}_{\,v}(G,\varphi).$$

Theorem 2.5. If $(G,\circ)$ and $(G,*)$ are two group structures on G then the operations $\circ$ and $*$ are reciprocally associative if and only if both groups are simultaneously n-extensible for every natural number $n \geq 3$.

Proof: If $(G,\circ)$ and $(G,*)$ are simultaneously *n*-extensible and $(G,\circ) = Red_{\,u}(G,\varphi)$, $(G,*) = \mathrm{Red}_{\,v}(G,\varphi)$, where (G,φ) is an *n*-group, then according to relations by Hosszú reduced groups (see [4]) we have:

$$x \circ (y * z) = (x \circ y) * z,\ x * (y \circ z) = (x * z) = (x * y) \circ z \text{, for all } x, y, z \in G,$$

so the operations $\circ$ and $*$ are reciprocally associable.

Conversely: From Remark 2.3, if the operations $\circ$ and $*$ are reciprocally associable, we have:

$$x * y = x \circ (u * u) \circ y = x \circ v^{-1} \circ y \text{ for } x, y \in G,$$

where *u* and *v* are the unit elements in $(G,\circ)$ and $(G,*)$, respectively.

For $n \geq 3$ we define on *G* the structure of *n*-group with the *n*-ary operation:

$$\varphi(x_1, x_2, ..., x_n) = x_1 \circ x_2 \circ ... \circ x_n, \text{ for all } x_1, x_2, ..., x_n \in G.$$

From the definition of the skew element $\overline{x}$ we have:

$$x = \varphi(x, ..., x, \overline{x}) = \varphi(\overline{x}, x, ..., x),$$

hence

$$x = x^{n-1} \circ \overline{x} = \overline{x} \circ x^{n-1},$$

$$\text{so } \overline{x} = x^{2-n} \text{ and from } u = u^{n-1} \circ \overline{u} \text{ it follows } \overline{u} = u.$$

We denote $(G, \top)$=*Red* $_u(G, \varphi)$ and $(G, \perp)$ = *Red* $_v(G, \varphi)$, where $v = (u * u)^{-1}$ (from Theorem 2.2).

We have

$$x \top y = \varphi(x, \underset{n-3}{u}, \overline{u}, y) = x \circ u^{n-3} \circ \overline{u} \circ y = x \circ u^{n-2} \circ y = x \circ y$$

and

$$x \perp y = \varphi(x, \underset{n-3}{v}, \overline{v}, y) = x \circ v^{n-3} \circ \overline{v} \circ y = x \circ v^{n-3} \circ v^{2-n} \circ y$$

$$= x \circ v^{-1} \circ y = x * y, \text{ for all } x, y \in G.$$

So Red $_u(G, \varphi) = (G,\circ)$ *and Red* $_v(G, \varphi) = (G,*)$.

Corollary 2.6. If $(G,\circ)$ is a group, then the binary operation $*$ on G is reciprocally associable with the operation $\circ$ if and only if for every $n \geq 3$ there exist an n-group (G, φ) and the elements $u, v \in G$ such that the binary operations have the form:

$$x \circ y = \varphi(x, \underset{n-3}{u}, \overline{u}, y), \; x, y \in G$$

$$x * y = \varphi(x, \underset{n-3}{v}, \overline{v}, y), \; x, y \in G$$

where $\overline{u}, \overline{v}$ are the skew elements of u, v.

Definition 2.7. If the binary operations $\circ$ and $*$ are reciprocally associable on the set *G*, then the ternary operation

$$[\circ,*] : G \times G \times G \to G$$

defined by

$$[\circ,*](x, y, z) = x \circ (y * z) \text{ for all } x, y, z \in G$$

is called the composed or justaposed operation of $\circ$ and $*$.

Let $(G,\circ)$ and $(G,*)$ be two group structures on G, u the unit element in $(G,\circ)$ and v the unit in $(G,*)$.

Theorem 2.8. If the group operations $\circ$ and $*$ are reciprocally associable on G, then the following statements are equivalent:

a) $v \circ x = x \circ v$, for all $x \in G$.

b) $u * x = x * u$, for all $x \in G$.

c) The ternary operation $[\circ,*]$ and $[*,\circ]$ coincide.

d) The ternary operation $[\circ,*]$ endows G with a 3-group structure.

Proof:

a) $\Rightarrow$ b) From Remark 2.3 we have $v = (u * u)^{-1}$; then

$$v \circ x = x \circ v \Leftrightarrow x \circ v^{-1} = v^{-1} \circ x \Leftrightarrow x \circ (u * u) = (u * u) \circ x \Leftrightarrow$$
$$(x \circ u) * u = u * (u \circ x) \Leftrightarrow x * u = u * x,\ x \in G.$$

b) $\Rightarrow$ c) We have

$$[\circ,*](x,y,z) = x \circ (y * z) = x \circ (y \circ v^{-1} \circ z) = x \circ y \circ z \circ v^{-1}$$

and

$$[x,\circ](x,y,z) = x * (y \circ z) = x \circ v^{-1} \circ (y \circ z) = x \circ y \circ z \circ v^{-1}$$

(from $v \circ x = x \circ v,\ x \in G$ it follows $v^{-1} \circ x = x \circ v^{-1},\ x \in G$).

c) $\Rightarrow$ d) We have $[\circ,*](x,y,z) = x \circ y \circ z \circ v^{-1}$. Since $(G,\circ)$ is a group, the applications $f, g, h : G \to G$ defined by

$$f(z) = a \circ b \circ z \circ v^{-1},\ z \in G$$
$$g(y) = a \circ y \circ b \circ v^{-1},\ y \in G$$
$$h(x) = x \circ a \circ b \circ v^{-1},\ x \in G$$

are bijective for all $a, b \in G$.

For the associativity we have:

$$[\circ,*]([\circ,*](x,y,z),x,t) = [\circ,*](x \circ y \circ z \circ v^{-1},x,t)$$
$$= x \circ y \circ z \circ v^{-1} \circ s \circ t \circ v^{-1} = x \circ y \circ z \circ s \circ t \circ v^{-1} \circ v^{-1}$$

and the same results for

$$[\circ,*](x,[\circ,*](y,z,s),t) \text{ and } [\circ,*](x,y,[\circ,x](z,s,t)).$$

So $(G,[\circ,*])$ is a 3-group.

d) $\Rightarrow$ a) From the associativity of the ternary operation we obtain

$$[\circ,*]([\circ,*](x,y,z),s,t) = [\circ,*](x,[\circ,*](y,z,s),t) \Leftrightarrow$$
$$x \circ y \circ z \circ v^{-1} \circ s \circ t \circ v^{-1} = x \circ y \circ z \circ s \circ v^{-1} \circ t \circ v^{-1} \Leftrightarrow$$
$$v^{-1} \circ s = s \circ v^{-1} \Leftrightarrow s \circ v = v \circ s, \text{ for every } s \in G.$$

Remark 2.9.

- The conditions a) and b) may be written as: $v \in Z(G,\circ)$ and $u \in Z(G,*)$ (the unit element of each group belongs to the center of the other group).
- The condition c) means that the operations $\circ$ and $*$ are commutable, in the sense of the following definition:

Definition 2.10. The binary operations $\circ$ and $*$ on the set G are commutable if the following relations hold:

$$x*(y\circ z)=(x\circ y)*z,\ x,y,z\in G$$
$$x\circ(y*z)=(x*y)\circ z,\ x,y,z\in G.$$

In [2] J. Dhombres gives conditions under which an operation which is commutable with a monoid operation, is associative.

Theorem 2.11. If $\circ$ and $*$ are reciprocally associable group operations on G then $(G,[\circ,*])$ is a 3-group if and only if the centers of the groups $Z(G,\circ)$ and $Z(G,*)$ coincide.

Proof: It is enough to show that the condition $Z(G,\circ)=Z(G,*)$ is equivalent to one of the conditions of Theorem 2.8. If $Z(G,\circ)=Z(G,*)$ then $v\in Z(G,*)$ implies $v\in Z(G,\circ)$ (the condition a) by Theorem 2.8).

If $v\in Z(G,\circ)$, from Remark 2.3 we have

$$x*y=x\circ v^{-1}\circ y=x\circ y\circ v^{-1},\ x,y\in G.$$

Then

$$x\in Z(G,*)\Leftrightarrow x*y=y*x,\ y\in G\ \Leftrightarrow\ x\circ y\circ v^{-1}=y\circ x\circ v^{-1},\ y\in G\Leftrightarrow$$
$$x\circ y=y\circ x,\ y\in G\Leftrightarrow x\in Z(G,\circ).$$

So $Z(G,*)=Z(G,\circ)$.

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