***ORIGINAL PAPER***

# INVOLUTE CURVES OF BIHARMONIC CURVES IN $\widetilde{SL_2(\mathbb{R})}$

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***Abstract.*** *In this paper, we study involute curves of biharmonic curves in the $\widetilde{SL_2(\mathbb{R})}$.Finally, we find out their explicit parametric equations.*

***Keywords:*** *Biharmonic curve, $\widetilde{SL_2(\mathbb{R})}$, curvature, torsion.*

## 1. INTRODUCTION

A smooth map $\phi : N \rightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |T(\phi)|^2 \, dv_h,$$

where $T(\phi) := tr\nabla^{\phi} d\phi$ is the tension field of $\phi$.

The Euler-Lagrange equation of the bienergy is given by $T_2(\phi) = 0$.
Here the section $T_2(\phi)$ is defined by:

$$T_2(\phi) = -\Delta_\phi T(\phi) + \mathrm{tr}R(T(\phi), d\phi)d\phi, \tag{1.1}$$

and called the bitension field of $\phi$. Non-harmonic biharmonic maps are called proper biharmonic maps. In this paper, we study involute curves of biharmonic curves in the $\widetilde{SL_2(\mathbb{R})}$. Finally, we find out their explicit parametric equations.

## 2. $\widetilde{SL_2(\mathbb{R})}$

We identify $\widetilde{SL_2(\mathbb{R})}$ with

$$\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\},$$

endowed with the metric

$$g = ds^2 = \left(dx + \frac{dy}{z}\right)^2 + \frac{dy^2 + dz^2}{z^2}.$$

The following set of left-invariant vector fields forms an orthonormal basis for $\widetilde{SL_2(\mathbb{R})}$

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$$e_1 = \frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y} - \frac{\partial}{\partial x}, e_3 = z\frac{\partial}{\partial z}. \tag{2.1}$$

The characterising properties of $g$deined by:

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1,$$
$$g(e_1,e_2) = g(e_2,e_3) = g(e_1,e_3) = 0.$$

The Riemannian connection $\nabla$ of the metric g is given by:

$$2g(\nabla_x Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)$$
$$- g(X,[Y,Z]) - g\ (Y,[X,Z]) + g(Z,[X,Y]),$$

which is known as Koszul's formula.

Using the Koszul's formula, we obtain:

$$\nabla_{e1}e_1 = 0, \quad \nabla_{e1}e_2 = \frac{1}{2}e_3, \quad \nabla_{e1}e_3 = -\frac{1}{2}e_2,$$
$$\nabla_{e2}e_1 = \frac{1}{2}e^3, \quad \nabla_{e2}e_2 = e_3, \quad \nabla_{e2}e_3 = -\frac{1}{2}e_1 - e_2, \tag{2.2}$$
$$\nabla_{e3}e_1 = -\frac{1}{2}e_2, \quad \nabla_{e3}e_2 = \frac{1}{2}e_1, \quad \nabla_{e3}e_3 = -0.$$

Moreover we put

$$R_{ijk} = (e_i, e_j)e_k, \ R_{ijkl} = (e_i, e_j, e_k, e_l),$$

where the indices $i$, $j$, $k$ and $l$ take the values 1, 2 and 3

$$R_{1212} = R_{1313} = \frac{1}{4}, \ R_{2323} = -\frac{7}{4}. \tag{2.3}$$

## 3. BIHARMONIC CURVES IN $\widetilde{SL_2(\mathbb{R})}$

Biharmonic equation for the curve $\gamma$ reduces to

$$\nabla_T^3 T - R(T, \nabla_T T)T = 0, \tag{3.1}$$

that is, $\gamma$ is called a biharmonic curve if it is a solution of the equation (3.1).

Let us consider biharmonicity of curves in $\widetilde{SL_2(\mathbb{R})}$.Let $\{T, N, B\}$ be the Frenet frame field along $\gamma$. Then, the Frenet frame satisfes the following Frenet-Serret equations:

$$\nabla_T T = \kappa N,$$
$$\nabla_T N = -\kappa T + \tau B, \quad (3.2)$$
$$\nabla_T B = -\tau N,$$

where $\kappa = |T(\gamma)| = |\nabla_T T|$ is the curvature of $\gamma$ and $\tau$ its torsion and

$$g(T,T) = 1,\ g(N,N) = 1,\ g(B,B) = 1,$$
$$g(T,N) = g(T,B) = g(N,B) = 0.$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$, we can write:

$$T = T_1 e_1 + T_2 e_2 + T_3 e_3,$$
$$N = N_1 e_1 + N_2 e_2 + N_3 e_3, \quad (3.3)$$
$$B = T \times N = B_1 e_1 + B_2 e_2 + B_3 e_3.$$

**Theorem 3.1.** $\gamma : I \to \widetilde{SL_2(\mathbb{R})}$ is a biharmonic curve if and only if:

$$\kappa = \text{constant} \neq 0,$$
$$\kappa^2 + \tau^2 = -\frac{1}{4} + \frac{15}{4} B_1^2, \quad (3.4)$$
$$\tau' = 2 N_1 B_{1.}$$

*Proof:* Using (4.1) and Frenet formulas (4.2), we have (4.4).

**Theorem 3.2.** ([9]) Let $\gamma : I \to \mathbb{R}$ be a unit speed non-geodesic curve. Then, the parametric equations of $\gamma$ are:

$$x(s) = \frac{1}{\aleph} sin\varphi sin[\aleph s + C] + \frac{1}{\aleph} sin\varphi cos[\aleph s + C] + \wp_2$$
$$y(s) = \frac{1}{\aleph 2 + cos^2\varphi} sin\varphi \wp_1 e^{cos\varphi s}\left(-\aleph cos[\aleph s + C] + cos\varphi sin[\aleph s + C]\right)$$
$$z(s) = \wp_1 e^{cos\varphi s}$$

where $\wp_1$, $\wp_2$ are constants of integration.

## 4. INVOLUTE CURVES OF BIHARMONIC CURVES IN $\widetilde{SL_2(\mathbb{R})}$

Definition 4.1. Let unit speed curve $\gamma : I \to \widetilde{SL_2(\mathbb{R})}$ and the curve $\Theta : I \to \widetilde{SL_2(\mathbb{R})}$ be given. For $\forall s \in I$ then the curve $\Theta$ is called the involute of the curve $\gamma$, if the tangent at the point $\gamma(s)$ to the curve $\gamma$ passes through the tangent at the point $\Theta(s)$ to the curve $\Theta$ and

$$g\left(T^*(s),T(s)\right)=0 \tag{4.1}$$

Let the Frenet-Serret frames of the curves $\gamma$ and $\Theta$ be $\{T,N,B\}$ and $\{T^*,N^*,B^*\}$, respectively.

**Theorem 4.2.** Let $\gamma : I \to \widetilde{SL_2(\mathbb{R})}$ be a unit speed biharmonic curve and $\Theta$ its involute curve on $\widetilde{SL_2(\mathbb{R})}$. Then, the parametric equations of $\Theta$ are

$$\begin{aligned}x(s)&=\frac{1}{\aleph}\sin\varphi\sin[\aleph s+C]+\frac{1}{\aleph}\sin\varphi\cos[\aleph s+C]+\\&+(\Lambda-s)\left(\sin\varphi\cos[\aleph s+C]-\sin\varphi\sin[\aleph s+C]+\wp_2\right)\end{aligned} \tag{4.2}$$

$$y(s)=\frac{1}{\aleph 2+\cos^2\varphi}\sin\varphi\wp_1 e^{\cos\varphi s}\left(-\aleph\cos[\aleph s+C]+\cos\varphi\sin[\aleph s+C]\right)+$$
$$+(\Lambda-s)\wp_1 e^{\cos\varphi s}\sin\varphi\sin[\aleph s+C]$$
$$z(s)=\wp_1 e^{\cos\varphi s}+(\Lambda-s)\wp_1 e^{\cos\varphi s}\cos\varphi$$

where $\Lambda$, C are constants of integration.

*Proof:* We assume that $\gamma : I \to \widetilde{SL_2(\mathbb{R})}$ be a unit speed biharmonic curve and $\Theta$ its involute curve on $\widetilde{SL_2(\mathbb{R})}$. We find that the parametric equations of $\Theta$.

The involute curve of biharmonic curve may be given as

$$\Theta(s)=\gamma(s)+\eta(s)T(s) \tag{4.3}$$

From (4.3), then we have

$$\Theta'(s)=\left(1+\eta'(s)\right)T(s)+\eta(s)\kappa(s)N(s) \tag{4.4}$$

Since the curve $\Theta$ is involute of the curve $\gamma$, $g\left(T^*(s),T(s)\right)=0$. Then, we get

$$1+\eta'(s)=0 \text{ or } \eta(s)=\Lambda-s \tag{4.5}$$

where $\Lambda$ is constant of integration.

Substituting (4.5) into (4.3), we get

$$\Theta(s)=\gamma(s)+(\Lambda-s)T(s) \tag{4.6}$$

On the other hand, the tangent vector can be written in the following form:

$$T=T_1e_1+T_2e_2+T_3e_3 \tag{4.7}$$

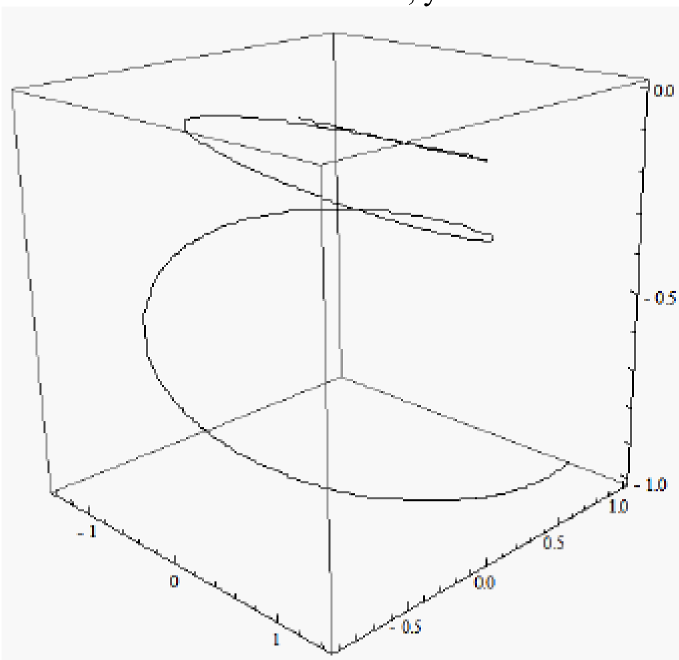
Using (2.1) in (3.5), we obtain

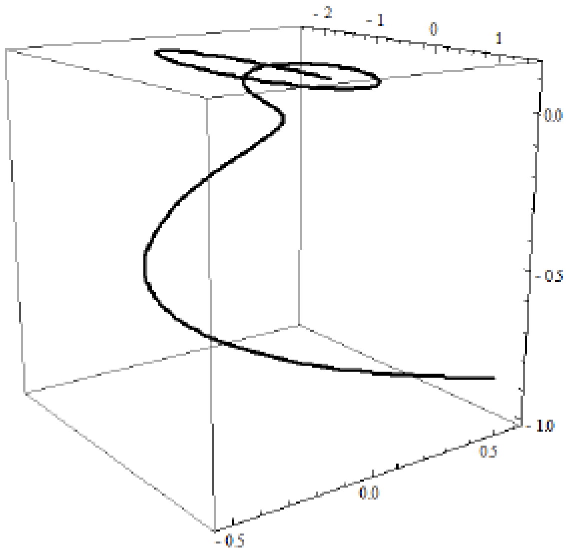
$$T = \left(\sin\varphi\cos\left[\aleph s + C\right] - \sin\varphi\sin\left[\aleph s + C\right],\right.$$
$$\left.\wp_1 e^{\cos\varphi s}\sin\varphi\sin\left[\aleph s + C\right], \wp_1 e^{\cos\varphi s}\cos\varphi\right) \quad (4.8)$$

If we substitute (4.10) into (4.6), we have (4.2). This concludes the proof of Theorem.

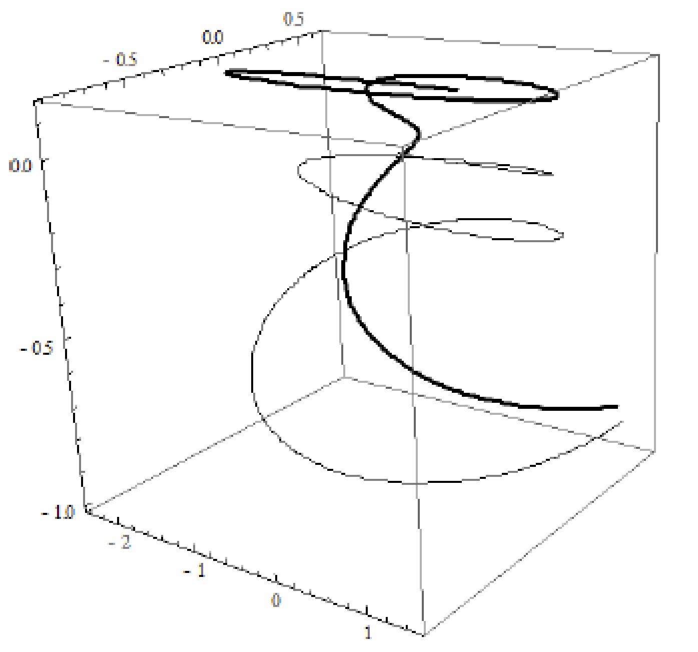
## 5. APPLICATIONS

We can use Mathematica in theorem 3.2-4.2 , yields




We show that γ and Θ in terms of Mathematica as follows:

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