ORIGINAL PAPER

STUDY OF THE FINITE FINAL POINT OPTIMAL CONTROL PROBLEM

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Abstract. *The paper presents a simple method for obtaining the analytical formula and the graphical representation of the trajectory and optimal control for a class of control problems, with fixed or free final state form.*

Keywords: *optimal, fix final point, solution.*

1. SOLVING THE FREE FINAL POINT OPTIMAL CONTROL PROBLEM

In this section we present a semi-inverse method for obtaining the analytical formula and the graphical representation of the trajectory and optimal control for a class of control problems. This method can be applied in Kalman's theory [1] to any pair of square matrices, *A(t)* and *B(t)*, symmetric and positive defined and for any initial input. The dynamical system Σ will have the following form:

$$\frac{dx}{dt} = f(x,u,t) = A(t)x + B(t)u \qquad (1)$$
$$y(t) = C(t)x$$

where : *x*= the state vector, *u*= the command(the vector of inputs), *y*=the vector of outputs, *t*=the time variable, *f(x, u, t)* is a self-adjoint operator (symmetrical), particularly, a symmetric, positive matrix, in the finite dimensional case (in the finite dimensional case, *A(t), B(t), C(t)* are also symmetric and positive matrices).

The Hilbert spaces that own these variables are: the state space *X*, the output space *Y*, the space of admitted commands *U* and Ω, the set of the regular and bounded functions defined on $(T_1, T_2) \in \mathrm{R}$, values in *U*.

Let φ and $\rho(t)$, two positive elements in *L(Y, Y)*, so that: $t \rightarrow \rho(t)$ is regular on (T_1, T_2) and let $\sigma(t)$, a positive element in *L(U, U)* so that the application $t \rightarrow \sigma(t)$ is continuous on (T_1, T_2). In the context of Kalman's theory, optimal control problem for the system Σ is limited to determine the *u(t)* command which minimizes the quadratic cost functional for any initial pair $(t_0, x_0) \in (T_1, T_2) \times X$. If $p : (T_1, T_2) \rightarrow L(X, X)$ is continuous differentiable then $u(t, x) = ½< x, p(x)x >$ is also continuous differentiable. According to Kalman's theory, if found a Hamilton Jacobi solution *u(t, x)*, then the optimal control problem can be solved. It shows that $p(\cdot)$ is the solution of a Riccati type equation formed with the Σ system coefficients.

$$\dot{p}(t) = -p^*(t)A(t) - A^*(t)p(t) + p^*(t)Q(t)p(t) + \rho(t) \qquad (2)$$

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where: $Q(t) = B(t) \cdot \sigma^{-1}(t) \cdot B^*(t)$, $t \rightarrow \rho(t)$ is a continuous application, $t \rightarrow \sigma(t)$ is regular on (T_1, T_2). For the system Σ, the Kalman's theory asserts:

Theorem 1.1. For any initial pair $(t_0, x_0) \subset (T_1, t_1] \times X$, with $t_1 \in (T_1, T_2)$, the optimal control problem for Σ has a solution.

- The optimal command is:

$$u_0(t, x) = -\sigma^{-1}(t) \cdot B^* \cdot p(t) \cdot x(t)$$

with: $p(\cdot)$ the unique solution of Riccati type equation:

$$\dot{p}(t) + p^*(t)A(t) + A^*(t)p(t) - p^*(t)Q(t)p(t) = \rho(t)$$

which verifies:

$$p(t_1) = \Psi(t_1) \cdot \Psi^{-1}(t_0) = e^{A \cdot (t1-t0)}$$

with: $S(t) = B(t) \cdot \sigma^{-1}(t) \cdot B^*(t)$

- The optimal trajectory is a solution of the linear equation:

$$\dot{x}(t) = [A - Q(t) \cdot p(t)] \cdot x(t)$$

with: $x(t_0) = x_0$.

Particularly:

Corollary 1.2. For any initial pair $(t_0, x_0) \subset (T_1, t_1) \times X$, with $t_1 \in (T_1, T_2)$, when ρ and σ , are assumed positive defined elements, so that $\rho(t) = 0$ and $\sigma(t) = 1$, the optimal control problem for Σ has a solution.

In order to apply the semi-inverse method, let's suppose that matrix $f(x,u,t)=A(t)\cdot x(t)+B(t)\cdot u(t)$ is symmetric, positive, and therefore the dynamical system is in the terms of Kalman's theory, with the initial restrictions:

$$A, B, C \in M_n(R), X = Y = R_n, t_0 = 0 < t_1, (t_0, t_1) \subset (T_1, T_2) \times R, x_0 = x(0),$$
$$p(t) = p^*(t), C_0 = e^A(t_0), Q = B(t) \cdot B^*(t).$$

In line with the previous corollary, the method refers only to the systems with $\rho(t) = 0$, $\sigma(t) = 1$, and constant coefficients, for which is presented the general analytical formula of the optimal trajectory and control. The "optimal trajectory" is analytical represented as follows:

$$x(t) = x_0 \exp[\int_0^t [A - Q \cdot p(\nu)] d\nu \tag{3}$$

as the solution of the linear equation: $\dot{x}(t) = [A - Qp(\nu)]x(\nu)$, with:

$$p(t) = [Q \cdot (A + A^*)^{-1} - e^t(A + A^*)^{-1}(-e^{-2A-A^*} + e^{-A-A^*} \cdot Q \cdot (A + A^*)^{-1})]^{-1} \tag{4}$$

and the optimal control is expressed by the next formula :

$$u_0(t, x) = -B^* \cdot [C_1 - e^{(A + A^*)^{-1} \cdot t} \cdot C_2]^{-1} \cdot x_0 \cdot \exp\left[\int_0^t [A - Q \cdot p(v)] dv\right] \tag{5}$$

with: $C_1 = Q \cdot (A + A^*)^{-1}$ and $C_2 = -e^{-2 \cdot A-A^*} + e^{-A-A^*} \cdot Q \cdot (A + A^*)^{-1}$. The method can be extended to any pair of square matrices, $A(t)$ and $B(t)$, depending on time, also to any systems Σ with self-adjoint operators like coefficients.

2. OPTIMAL CONTROL THEORY WITH FINAL FIXED POINT

Suppose the optimal control problem for system Σ and the assumptions in the first section. We are now interested about "the quadratic cost functional" denoted by J, having K and L as its "terms". If $K(t, x)$ is:

$$K(t,x)=\frac{1}{2}\|x\|_{\varphi}^{2} \tag{6}$$

and $L(x, u, t)$ is:

$$L(x,u,t)=\frac{1}{2}[\|x\|_{\rho(t)}^{2}+\|u\|_{\sigma(t)}^{2}] \tag{7}$$

then the cost functional $J(t_0, x_0, u(\cdot))$ expressed with aid of K and L, is called "the quadratic cost functional" for the linear system Σ.

K is the terminal cost and respectively, L is the cost associated to the trajectory, when $t_1 > t_0$, t_0, $t_1 \in X$ are fixed. Also: $S = \{t_1\} \times X$ is the target set(consequently, the set of the admitted states) As the time t_1 is also fixed and the final state is free, it occurs that every $u(\cdot) \in \Omega$ will transfer (t_0, x_0) in S, consequently:

$$J(t_0,x_0,u(\cdot))=K(t_1,x_u)+\int_{t_0}^{t_1}L(x_u,u,t)dt \tag{8}$$

with: $x_u(t) = \varphi(t;\, t_0, x_0, u(\cdot))$ and the optimization problem is reduced to the finding of $u(\cdot)$ which minimizes the cost $J(t_0, x_0, u(\cdot))$.

Consider now an arbitrary normed linear space B, with a differentiable norm, a nonempty subset $\Phi \subset B$, a real functional, $J : B \to R$, and the problem of minimization for J on Φ. We call "minimizing convergent family" for J, $\{v_\delta \mid v_\delta \in B\}$, if the next two conditions are verified [1] :

- $\lim\limits_{\delta\to\infty} J(v_\delta)=J^0=\inf\limits_{\Phi} J(\cdot)$
- $\lim\limits_{\delta\to\infty} v_\delta = v^0$

Let's also suppose J is smooth enough so that:

$$J(v^0)=J(\lim_{\delta\to\infty} v_\delta)=\lim_{\delta\to\infty}(J(v_\delta))=J^0$$

According to Weierstrass theorem [2], it is known that if J is lower semi-continuous or continuous in v_0 then v_0 is minimizing point for J on Φ:

$$J(v^0)=J(\lim_{\delta\to\infty} v_\delta)=\lim_{\delta\to\infty} J(v_\delta)=J^0=\inf_{\Phi}\{J(v)\}$$

These considerations are useful if studying the optimality problem which can be reduced to the minimization of the cost functional.

Theorem 2.1. Let's suppose that J is lower semi-continuous relating to F (topology of space B) and there is a subset unbounded to right, $\delta \in D \subset [0, \infty)$, with the following properties:

- $\delta \in D$ implies there is $v_\delta \in B$ so that: $J(v_\delta)+\delta\cdot\varphi(v_\delta) \le J(v)+\delta\cdot\varphi(v),\ \forall v \in B$
- There is $v_0 \in B$ so that: $\lim\limits_{\delta\to\infty} v_\delta = v^0$

then v_0 is an element in Φ and $J(v^0) = J^0 = \inf_{\Phi}\{J(v)\}$, consequently, v_0 is a solution of the considered problem.

The above theorem [1] justification is given by Courant [3]. The main significance of this theorem in the control problems, is to create the possibility of replacing the limit conditions with "free" problems. According to Kalman's theory, an optimal command leads to a Hamiltonian extreme, when the out state is free, but there are some difficulties if the final state is fixed. In this case, we apply the theorem as follows.

It is assumed that the target set is $t_1 \times x_1$ with t_1 and $x_1 = 0$ (or fixed), and

$$(u(\cdot)) = \frac{1}{2}\|x(t_1)\|^2$$

and we are going to find the initial conditions for the free final point problem which the cost functional is: $J(u(\cdot)) + \delta \cdot \varphi(u(\cdot))$. If there is a convenient convergent family of solutions for these free final point problems, we can get the necessary conditions for the final free point initial problem. Consequently, the author is led to the next corollary [4] (in the particular case, for $x_1 = 0$):

Corollary 2.2. Let's suppose that J is lower semi-continuous and there is a subset unbounded to right $D \subset [0, \infty)$ with the following properties:

1. $\delta \in D$ leads to the existence of $u_\delta(\cdot) \in \Omega$ so that:

$$J(u_\delta(\cdot)) + \frac{\delta}{2}\|u_\delta(t_1)\| \le J(u(\cdot)) + \frac{\delta}{2}\|u(t_1)\|, \quad \forall u(\cdot) \in \Omega \tag{9}$$

2. $\exists m \in [0,\infty)$ so that:

$$\lim_{\delta\to\infty} \delta \cdot \|x_{u_\delta}(t_1)\| = m \tag{10}$$

3. $\exists u^0(\cdot) \in \Omega$ so that:

$$\lim_{\delta\to\infty} u_\delta(\cdot) = u^0(\cdot) \tag{11}$$

(norm topology on Ω)

Then $u^0(\cdot)$ is an optimal command and there is an adjoint vector $\lambda^0(\cdot)$ corresponding to $u^0(\cdot)$ so that:

$$\frac{\partial H}{\partial u}(x^0(\cdot), \lambda^0(\cdot), u^0(\cdot)) = 0$$

almost everywhere in $[t_0, t_1]$.

We can notice that $u_\delta(\cdot)$ represents a convergent inferior set for u, consequently, also for J. The main difficulty in applications is to prove the conditions (10) and (11). Let's consider the scalar system [1] :

$$\dot{x} = -x + u, \; x(0) = x_0 \neq 0$$

the target set $S = T \times 0$ with T fixed and the cost functional J:

$$J(u) = \frac{1}{2}\int_0^T (x^2 + u^2)dt$$

In order to apply the above corollary, we assume the target set is $S = T \times R$ and the corresponding cost functional is the following, thus [4]:

$$J(u) = \frac{\delta}{2}x^2(T) + \frac{1}{2}\int_0^T (x^2 + u^2)dt \qquad \text{, with } \delta \in (0, \infty).$$

Last problem has the solution [4] :

$$u_\delta(t) = -p_\delta(t) \cdot x_\delta(t) \tag{12}$$

where $p_\delta(t)$ is the solution of a convenient Riccati type equation, formed with the scalar system coefficients and it is demonstrated that the last two conditions of the above corollary are verified.

3. OPTIMAL CONTROL THEORY FOR THE FINAL FIXED POINT, USING THE SEMI-INVERSE METHOD

3.1. FINDING SOLUTIONS FOR THE FINAL FIXED POINT CONTROL PROBLEM

Starting from the above assumptions, we shall analyze the same scalar system but with the modified cost functional:

$$J(u) = \frac{\delta}{2} x^2(T) + \frac{1}{2}\int_{t_0}^{T} u^2 dt$$

where: $\delta \in (0, \infty)$, $t_0 < T$ and the target set is $S = T \times R$. Let $\rho(t) = 0$ and $\sigma(t) = 1$. According to the semi-inverse method, we are led to the next Bernoulli type equation:

$$\dot{p}_\delta(t) - 2 \cdot p_\delta(t) = p_\delta^{\,2}(t) \tag{13}$$

with the solution:

$$p_\delta(t) = -\frac{e^{-2(t_0 - t)}}{\dfrac{3 + \ln(\delta)}{2(1 + \ln(\delta))} - \left(\dfrac{1}{2} - \dfrac{1}{2} e^{2(t - t_0)}\right)}$$

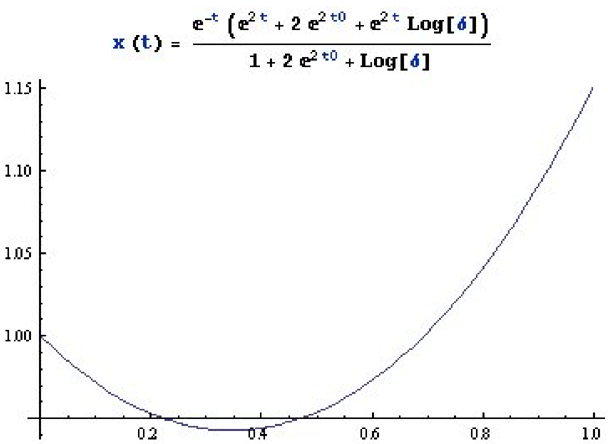

Fig. 1. System optimal trajectory for: T_0=0, δ=1.

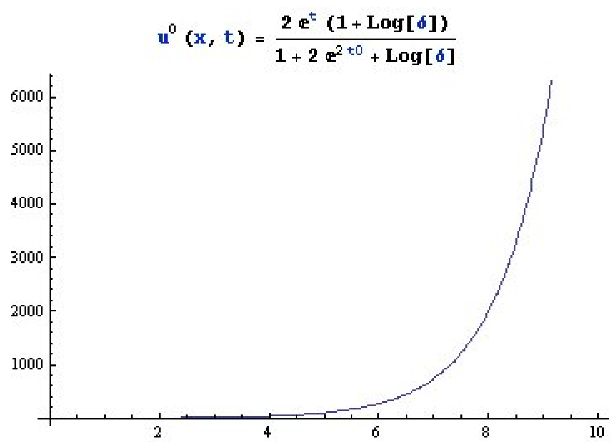

Fig. 2. System optimal trajectory for: T_0=0, δ=1.

and x_δ is expressed by the following formula:

$$x_\delta = x_0 \cdot e^{\int_{t_0}^{t} [-1 - p_\delta(\tau)] d\tau}$$

with: $t \in [t_0, t_1] \subset [0, T]$, which leads to:

$$x_\delta(t) = x_0 \cdot \frac{e^{-t}(2e^{2t_0} + e^{2t} + e^{2t}\ln(\delta))}{1 + 2e^{2t_0} + \ln(\delta)} \qquad \text{(See Fig. 1)}$$

and the corresponding optimal command:

$$u_\delta^0 = x_0 \cdot \frac{e^{2(t - t_0) - t}(2e^{2t_0} + e^{2t} + e^{2t}\ln(\delta))}{(1 + 2e^{2t_0} + \ln(\delta))\left(\dfrac{3 + \ln(\delta)}{2(1 + \ln(\delta))} - \dfrac{1}{2}(1 + 2e^{2t_0})\right)} \qquad \text{(See Fig. 2)}$$

3.2. A THEOREM FOR THE FIXED FINAL POINT CONTROL PROBLEM

Consequently, the Corollary 2.2 can be modified in the following form:

Theorem 3.1. Suppose that system *Σ* is in the conditions of the semi-inverse method and *J* is the quadratic cost functional. Let $D \subset [0, \infty)$ a subset unbounded to right, with the following properties:

1. $\delta \in D$ leads to the existence of $u_\delta(\cdot) \in \Omega$ so that:

$$J(u_\delta(\cdot)) = \frac{\delta}{2}\left\|x_{u_\delta}(t_1)\right\|^2 + \int_{t_0}^{t_1} u_\delta{}^2(t)dt \le \frac{\delta}{2}\left\|x_u(t_1)\right\|^2 + \int_{t_0}^{t_1} u^2(t)dt = J(u(\cdot)),\ \forall u(\cdot) \in \Omega \tag{14}$$

2. $\exists m \in [0,\infty)$ so that:

$$\lim_{\delta\to\infty} x_{u_\delta} = m \tag{15}$$

3. $\exists u^0(\cdot) \in \Omega$ so that:

$$\lim_{\delta\to\infty} u_\delta(\cdot) = u^0(\cdot) \tag{11}$$

(norm topology on **Ω**, when norm is supposed differentiable)

Then $u^0(\cdot)$ is an optimal command and there is an adjoint vector $\lambda^0(\cdot)$ corresponding to $u^0(\cdot)$ so that:

$$\frac{\partial H}{\partial u}(x^0(\cdot),\lambda^0(\cdot),u^0(\cdot)) = 0$$

almost everywhere in $[t_0, t_1] \subset [0, T] \subset R$

Proof: The optimality of the $u^0(\cdot)$ results from the main theorem. As:

$$H(x,\lambda,u,t) = \frac{1}{2}\|u\|^2 + < \lambda, A(t)\cdot x + B(t)\cdot u >$$

then $\frac{\partial H}{\partial u}$ is continuous. Since $\lim_{\delta\to\infty} u_\delta(\cdot) = u^0(\cdot)$ then: $\lim_{\delta\to\infty} x_\delta(\cdot) = x^0(\cdot)$.

It remains to prove that:

$$\lim_{\delta\to\infty} \lambda_\delta(\cdot) = \lambda^0(\cdot)$$

also that λ_δ is a solution of the differential equation: $\dot{\lambda} = -\left(\left.\frac{\partial f}{\partial x}\right|_0\right)^* \cdot \lambda - \left.\frac{\partial L}{\partial x}\right|_0$, where $\lambda\delta$ is the adjoint vector corresponding to $u_\delta(\cdot)$ so that:

$$\frac{\partial H}{\partial u}(x^0(\cdot),\lambda^0(\cdot),u^0(\cdot)) = 0$$

almost everywhere in $[t_0, t_1]$. According to Kalman's theory [1, ξ 3.5], the existence of $\lambda_\delta(\cdot)$ is proved, since $u_\delta(\cdot)$ is the optimal command for a free final point control problem.

Let's denote: $\lambda^0(\cdot)$ the solution for the previous equation, which satisfies the final restriction:

$$\lim_{\delta\to\infty} \lambda_\delta(t_1) = \lambda_1$$

$$\lambda_1 = \delta\cdot(x_{u\delta}(t_1) - x_1)$$

As a result of the $\frac{\partial L}{\partial x}$ and $\frac{\partial f}{\partial x}$ continuity, we are led to the following:

$$\lim_{\delta\to\infty}\left.\frac{\partial L}{\partial x}\right|_{\delta} = \left.\frac{\partial L}{\partial x}\right|_{0}$$

and:

$$\lim_{\delta\to\infty}\left.\frac{\partial f}{\partial x}\right|_{\delta} = \left.\frac{\partial f}{\partial x}\right|_{0}$$

Consequently, the two last conditions in the corollary are verified. Particularly, in the previous example, the last conditions are also verified:

1. $\lim_{\delta\to\infty} x_{u_\delta} = x_0 \cdot c_0 \cdot e^t \le m = x_0 \cdot e^t < \infty$, $t \in [t_0, t_1] \subseteq [0, T]$
2. $\lim_{\delta\to\infty} u_\delta(\cdot) = 2x_0 \cdot c_0 \cdot e^t$, $t \in [t_0, t_1] \subseteq [0, T]$

In terms of the last theorem, the dynamical system Σ can be written in the state variables and the optimal command can still be obtained in an analytical form ($\forall\delta \ge 0$).

4. CONCLUSIONS

From the above considerations, we can conclude that the semi-inverse method used for solving the free final state control problem, can be also applied for solving the optimal control problem with the fixed final point.

Consequently, the semi-inverse method can be considered as a general method for solving the free or fixed final state control problems, when $\rho(t) = 0$ and $\sigma(t) = 1$ restrictions in eq. (2) are verified for the dynamic system Σ.

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