ORIGINAL PAPER

ON A CLASS OF SUBMANIFOLDS OF A KENMOTSU MANIFOLD

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Abstract. *In this paper we study some properties of a class of normal anti-invariant submanifolds in Kenmotsu manifolds, called the class of Whitney type spheres. We also prove that Whitney type spheres have a closed conformal vector field in Kenmotsu space forms.*
Keywords: *Whitney type sphere, closed conformal vector field, Kenmotsu manifold.*

1. INTRODUCTION

The Whitney spheres were introduced in complex Euclidean space $\mathbb{C}^n$ [7, 8] as a family of Lagrangian immersions of the unit sphere S^n, whose second fundamental form satisfies a certain equality. In [7] Antonio Ros and Francisco Urbano studied Lagrangian submanifolds of $\mathbb{C}^n$ with conformal Maslov form and Whitney spheres. Later in [5] a class of Legendrian submanifolds with closed conformal vector field in Sasakian space forms was studied, where it was analyzed the existence of these submanifolds and were given their geometric properties. In this paper, we introduce a class of normal anti-invariant in Kenmotsu manifolds, called the Whitney type sphere through an analogy of those in the complex case and we prove some properties of them and that they have a closed conformal vector field in Kenmotsu space forms.

We remember some necessary useful notions and results for our next considerations.

Let $\tilde{M}$ be a C^∞- differentiabile, 2n+1 dimensional almost contact manifold with the almost contact metric structure (F, ξ, η, g), where F is a (1,1) tensor field, η is the Reeb vector field, all these tensors satisfying the following conditions:

$$F^2 = -I + \eta \otimes \xi; \quad \eta(\xi) = 1; \quad g(FX, FY) = g(X, Y) - \eta(X)\eta(Y), \tag{1}$$

for all X, Y in $\chi(\tilde{M})$, where $\chi(\tilde{M})$ represents the set of all vector fields on $\tilde{M}$.

Let M be a submanifold of $\tilde{M}$. We consider $\tilde{\nabla}$ the Levi-Civita connection on $\tilde{M}$, ∇ the Levi-Civita connection induced by $\tilde{\nabla}$ on M, $\nabla^{\perp}$ the connection in the normal bundle $T^{\perp}M$, h the second fundamental form on M and $A_{\vec{n}}$ the Weingarten operator. The well-known Gauss- Weingarten formulas on M are:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y); \quad \nabla_X \vec{n} = -A_{\vec{n}} X + \nabla^{\perp}_X \vec{n} \tag{2}$$

for X, Y in χ(M) and $\vec{n}$ in $\chi^{\perp}(M)$.

We consider the Sasaki form Ω on $\tilde{M}$, given by $\Omega(X, Y) = g(X, FY)$. Also, we denote by N_F the Nijenhius tensor of F and $N^{(1)} = N_F + 2d\eta \otimes \xi$. An almost contact manifold $\tilde{M}$ is a *Kenmotsu manifold* if

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$$d\eta = 0; \quad d\Omega = \eta \wedge \Omega; \quad N^{(1)} = 0.$$

It is also known that an almost contact manifold $\tilde{M}$ is a Kenmotsu manifold if and only if

$$(\tilde{\nabla}_X F)Y = -g(X, FY)\xi - \eta(X)FY; \quad \tilde{\nabla}_X \xi = X - \eta(X)\xi , \tag{3}$$

for all X, Y in $\chi(\tilde{M})$.

From [3] we have the following expression of the curvature tensor in Kenmotsu space forms:

$$\tilde{R}(X,Y)Z = \frac{c-3}{4}[g(Y,Z)X - g(X,Z)Y] + \frac{c+1}{4}[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X +$$
$$+g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \Omega(X,Z)FY - \Omega(Y,Z)FX + 2\Omega(X,Y)FZ]. \tag{4}$$

From [4], we recall the following:

Definition 1.1. A submanifold M of a Kenmotsu manifold $\tilde{M}$ is a *normal semi-invariant submanifold* if ξ is normal to M and M has two distributions D, $D^{\perp}$, called the invariant, respectively, the anti-invariant distribution of M, so that:

(i) $T_x M = D_x \oplus D_x^{\perp} \oplus \langle \xi_x \rangle$;

(ii) D_x, $D_x^{\perp}$, $\langle \xi_x \rangle$ are orthogonal;

(iii) $FD_x = D_x$, $FD_x^{\perp} \subset T_x^{\perp} M$,

for all x in M.

If $D = 0$ then M is *a normal anti-invariant submanifold* of $\tilde{M}$ and if $D^{\perp} = 0$ the M is *a normal invariant submanifold* of $\tilde{M}$.

Also, in [4], it was proved the following:

Lema 1.2. If M is a normal semi-invariant submanifold of a Kenmotsu manifold $\tilde{M}$, then:

(i) $A_{FX} Y = A_{FY} X$, for all $X, Y \in D^{\perp}$;

(ii) $A_\xi Z = -Z$ and $\nabla_Z^{\perp} \xi = 0$, for all $Z \in \chi(M)$.

2. RESULTS AND DISCUSSION

Proposition 2.1. If M is a m-dimensional, normal anti-invariant submanifold of a 2n+1-dimensional Kenmotsu manifold $\tilde{M}$, then $m \leq n$.

Proof: For $x \in M$ we have $T_x \tilde{M} = T_x M \oplus T_x^{\perp} M$ and dim $FT_x M$= dim $T_x M$= m. Moreover, because M is normal anti-invariant, we have $FT_x M \subset T_x^{\perp} M$; $FT_x M \perp \langle \xi_x \rangle$ and then dim$T_x^{\perp} M \geq$ dim $FT_x M +$dim$\langle \xi_x \rangle$=m+1. Now 2m≤m+dim $T_x^{\perp} M - 1 =$dim $T_x M +$dim$T_x^{\perp} M$-1 =dim $T_x \tilde{M}$-1=2n or m≤n.

For our next considerations, let M be a normal anti-invariant submanifold in a Kenmotsu manifold $\tilde{M}$. We have the following decomposition:

$$T_x^{\perp} M = FT_x M \oplus \tau_x M \oplus \langle \xi_x \rangle,$$

where $\tau_x M$ is the orthogonal complement of $FT_x M \oplus \langle \xi_x \rangle$ in $T_x^{\perp} M$, $FTM = \bigcup_{x \in M} FT_x M$, $\tau(M) = \bigcup_{x \in M} \tau_x M$. Let $h^{\tau}(X,Y)$, $h^F(X,Y)$ be the components of h(X,Y) on $\tau(M)$, respectively, FTM, for $X, Y \in \chi(M)$.

Propositon 2.2. Let M be a m-dimensional normal anti-invariant submanifold in a 2n+1-dimensional Kenmotsu manifold $\tilde{M}$. Then:

$$\nabla^{\perp} o F - F o \nabla + F o h^{\xi}; \; F o h^{\xi} + A_F - 0; \; h - h^F + h^{\xi} - g \otimes \xi. \tag{5}$$

Proof: From(3), Lema 1.2 *(ii)* and (2) we have

$$A_{\xi} X = -X; \;\; g(h(X,Y),\xi) = g(A_{\xi} X, Y) = -g(X,Y)$$

and then we obtain the last relation of (5). The first two relations of (5) follow easily using the fact that M is a normal anti-invariant submanifold, (2) and the first relation of (3).

We consider the 1-form:

$$\alpha_{\vec{n}} : \chi(M) \to F(M), \; \alpha_{\vec{n}}(Y) = g(Y, F\vec{n}), \forall \, Y \in \chi(M), \tag{6}$$

for all $\vec{n} \in \chi^{\perp}(M)$. In [2], it was proved that $\alpha_{\vec{n}}$ is closed if and only if

$$g(\nabla^{\perp}_{X}\vec{n}, FY) = g(\nabla^{\perp}_{Y}\vec{n}, FX), \forall \, X, Y \in \chi(M); \;\; \alpha_{\xi} = 0. \tag{7}$$

Definition 2.3. Let M be a normal anti-invariant submanifold of a Kenmotsu manifold $\tilde{M}$. A vector field $X \in \chi(M)$ is a *closed vector field* if the 1-form α_{FX} is closed.

Proposition 2.4. Let M be a m-dimensional normal anti-invariant submanifold of a 2n+1-dimensional Kenmotsu manifold $\tilde{M}$. If there is satisfied at least one of the following relations:

(i) the mean vector field H of M is parallel;

(ii) m=n,

then FH is a closed vector field.

Proof:

(i) because H is parallel, from (7) we obtain that α_H is closed and $\alpha_{F^2H} = \alpha_{-H+\eta(H)\xi} = -\alpha_H$, that is FH is closed.

(ii) m=n implies that $H = H^F$ and $\alpha_{FH}(X) = 0$, for all $X \in \chi(M)$.

Definition 2.5. Let M be a normal anti-invariant submanifold of a Kenmotsu manifold $\tilde{M}$. A vector field on M is a *closed conformal vector field* if

(i) X is closed

(ii) X is conformal, that is

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = \frac{2}{\dim M} div(X), \forall \, X, Y, Z \in \chi(M), \tag{8}$$

where div(X) represents the divergence of X. Moreover, if there is $f \in F(M)$ so that $h(X,X) = fFX$, then X is a *special closed conformal vector field.*

From [7], the conditions (i) and (ii) of Definition 2.5, are equivalent with:

$$\nabla_X Y = \frac{div(X)}{\dim M} Y, \forall \, Y \in \chi(M). \tag{9}$$

Definition 2.6. A n-dimensional normal anti-invariant submanifold of a 2n+1-dimensional Kenmotsu manifold $\tilde{M}$ is a *Whitney type sphere* if its fundamental form *h* satisfies the following relation:

$$h(V,V) = \lambda[g(V,V) + 2g(H,FV)FV + \frac{2}{n}g(H,\xi)g(V,V)\xi],\ \forall V \epsilon \chi(M), \tag{10}$$

where H is the mean curvature vector of M and $\lambda \epsilon F(M)$.

Proposition 2.7. If M is a Whitney type sphere of a 2n+1-dimensional Kenmotsu manifold $\tilde{M}$, then

$$\lambda = \frac{n}{n+2},\ \|H\| \geq 1,$$

and

$$h(V,W) = \lambda\left[g(V,W)H + g(H,FV)FW + g(H,FW)FV - \frac{2}{n}g(V,W)\xi\right], \tag{11}$$

for all $V,W \epsilon \chi(M)$, where λ is defined by (10) and $\|U\|$ represents the norm of $U \epsilon \chi(\tilde{M})$.

Proof: From Lema 1.2 (ii), we have $g(h(X,Y),\xi) = -g(X,Y),\ \forall X,Y \epsilon \chi(M)$.

Let $\{e_1,\dots,e_n,Fe_1,\dots,Fe_n,\xi\}$ be a F-basis of $\chi(\tilde{M})$ so that $\{e_1,\dots,e_n\}$ is a local orthonormal basis of $\chi(M)$. Then $g(H,\xi) = 1$ and $\|H\|^2 = \sum_{i=1}^{n} g^2(H,Fe_i) + 1$ or $\|H\| \geq 1$.

Now, from (10) we have $h(e_i,e_i) = \lambda[H + 2h(H,Fe_i)Fe_i + \frac{2}{n}g(H,\xi)\xi]$ and $nH = \sum_{i=1}^{n} h(e_i,e_i) = \lambda(n+2)H$ and then $\lambda = \frac{n}{n+2}$. Relation (11) results easily from the symmetry and bi-linearity of *h*.

Theorem 2.8. If M is an Whitney type sphere of a Kenmotsu space form $\tilde{M}(c)$, then FH is a closed conformal vector field of M.

Proof:

For $Z,V \epsilon \chi(M)$ and *V* an unitary vector field $V \perp Z$, from (4) we have

$$\tilde{R}(Z,V)V = \frac{c-3}{4}Z$$

and then $[\tilde{R}(Z,V)V]^{\perp} = 0$. Now, from the Codazzi equation, we obtain

$$(\overline{\nabla}_Z^{\perp} h)(V,V) = (\overline{\nabla}_V^{\perp} h)(Z,V)$$

and from the properties of Levi-Civita connexion, Lema 1.2 (ii), (3), (5) and (11), it results that

$$(\overline{\nabla}_Z^{\perp} h)(V,V) = \lambda[\nabla_Z^{\perp} H) + 2g(\nabla_Z^{\perp} H,FV)FV]$$

and

$$(\overline{\nabla}_V^{\perp} h)(Z,V) = \lambda[g(\nabla_V^{\perp} H,FV)FZ + g(\nabla_V^{\perp} H,FZ)FV].$$

From these last two relations, we obtain:

$$\nabla_Z^{\perp} H + 2g(\nabla_Z^{\perp} H,FV)FV = g(\nabla_V^{\perp} H,FV)FZ + g(\nabla_V^{\perp} H,FZ)FV. \tag{12}$$

From (12), because V is unitary, we have

$$g(\nabla_Z^{\perp} H,FV) + 2g(FV,\nabla_Z^{\perp} H) = g(\nabla_V^{\perp} H,FZ). \tag{13}$$

Because M is an n-dimensional normal anti-invariant submanifold, using Proposition 2.4 (ii), it results that FH is a closed vector field. Moreover, we have $g(FV,\nabla_Z^{\perp} H) = g(FZ,\nabla_V^{\perp} H)$ and from (13) it results that

$$g(FV,\nabla_Z^{\perp} H) = 0. \tag{14}$$

From (2) and (11), we have:

$$\nabla_Z(FH) = -\eta(H)FZ - g(Z,FH)\xi + F\nabla_Z^{\perp}H - FA_H Z$$
$$-\lambda[g(Z,FH)H - g(FZ,H)H - g(FZ,H)\xi - \|H\|^2 FZ + FZ],$$

and then

$$\nabla_Z(FH) = F\nabla_Z^{\perp}H. \qquad (15)$$

From (12) and (15), it results that $\nabla_Z(FH) = -g(FV, \nabla_V^{\perp}H)Z$ and then, using (9) we obtain that FH is a closed conformal vector field.

Proposition 2.9. Let M be an Whitney type sphere of a 2n+1-dimensional Kenmotsu space form $\tilde{M}(c)$. Then the Ricci tensor Ric, the scalar curvature ρ and the sectional curvature K of M have the following expressions:

(i) $Ric(V) = \left[\frac{(n-1)(c-3)}{4} + \frac{3n^2-4}{(n+1)^2}\right]\|V\|^2 + \frac{n^2}{(n+1)^2}[n\|V\|^2\|H\|^2 + (n-2)g^2(FV,H);$ (16)

(ii) $\rho = \frac{n(n-1)(c-3)}{4} + \frac{2n(n-1)}{(n+2)} + \frac{n^2(n-1)}{(n+2)}\|H\|^2;$ (17)

(iii) $K(X,Y) = \frac{(c-3)}{4} + \frac{n^2}{(n+1)^2}\left[\|H\|^2 + 3g^2(H,FX) + 3g^2(H,FY) + \frac{4}{n} + \frac{4}{n^2}\right],$ (18)

where X, Y are two orthonormal vector fields on M.

Proof: Let $\{e_1, \dots, e_n\}$ be a local orthonormal basis of M. From Gauss equation, we have:

$$Ric(V) = \sum_{i=1}^{n} g(\tilde{R}(e_i,V)V,e_i) + \sum_{i=1}^{n} g(h(e_i,e_i),h(V,V)) - \sum_{i=1}^{n} g(h(e_i,V),h(e_i,V)).$$

From (4) it results that

$$\sum_{i=1}^{n} g(\tilde{R}(e_i,V)V,e_i) = \frac{(c-3)(n-1)}{4}[n\|V\|^2 - g(V, \sum_{i=1}^{n} g(e_i,V)e_i) = \frac{c-3}{4}(n-1)\|H\|^2$$

and from (10), we have

$$\sum_{i=1}^{n} g(h(e_i,e_i),h(V,V)) = ng(H,h(V,V)) = n\lambda[\|H\|^2 + 2g^2(H,FV) + \frac{2}{n}\|H\|^2];$$

$$\sum_{i=1}^{n} g(h(e_i,V),h(e_i,V)) = \frac{n^2}{(n+1)^2}[2\|H\|^2\|V\|^2 + 6g^2(FV,H) + (\tfrac{4}{n}+\tfrac{4}{n^2} - 1)\|V\|^2$$
$$+ng^2(H,FV)].$$

From these last four relations we obtain (16).

(i) results from (i) and the fact that $\rho = \sum_{i=1}^{n} Ric(e_i)$.

(ii) from the Gauss equation, we have

$$K(X,Y) = g(R(X,Y)Y,X) = g(\tilde{R}(X,Y)Y,X) + g(h(X,X),h(Y,Y)) - g(h(X,Y),h(X,Y)),$$

for X, Y two orthonormal vector fields on M,

Because M is a normal anti-invariant submanifold, from (4) we have $\tilde{R}(X,Y)Y = \frac{c-3}{4}X;$

$$g(h(X,X),h(Y,Y)) = \lambda^2[\|H\|^2 + 2g^2(H,FY) + 2g^2(H,FX) + \frac{4}{n} + \frac{4}{n^2}]$$

and

$$g\big(h(X,Y),h(X,Y)\big) = \lambda^2[g^2(H,FX) + g^2(H,FX)].$$

From these four relations we obtain (18).

Proposition 2.10. If M is an Whitney type sphere of a 2n+1-dimensional Kenmotsu space form $\tilde{M}(c)$, with H the mean curvature vector of M, Ric the Ricci tensor field and K the sectional curvature of M, then:

$$\frac{(n-1)(c-3)}{4} + \frac{3n^2-4}{(n+2)^2} + \frac{n^2}{(n+2)^2}\|H\|^2 \le Ric \le \frac{(n-1)(c-3)}{4} + \frac{3n^2-4}{(n+2)^2} + \frac{2n^2(n-1)}{(n+2)}\|H\|^2;$$

$$K(X,Y) \ge \frac{c-3}{4} + \frac{4n^2}{(n+2)^2}\Big(\|H\|^2 + \frac{1}{n} + \frac{1}{n^2}\Big),$$

where X, Y are two orthonormal vector fields on M.

Proof: it results from Proposition 2.9 and the Cauchy's inegality $g^2(U,V) \le g(U,U)g(V,V)$.

Theorem 2.11. There are not compact Whitney type spheres on a Kenmotsu space form $\tilde{M}(c)$.

Proof: Suppose that M is a compact Whitney type sphere in the Kenmotsu space form $\tilde{M}(c)$. From Bochner's Theorem, the first group of Rham cohomology is $H^1(M;R) = 0$. Because M is an Whitney type sphere, from Proposition 2.4 (ii), we have that FH is a closed vector field and there is $f \in F(M)$ so that $FH = grad(f)$. Moreover, because M is compact, it results that f has critical points which are zeroes of H and this is in contradictory with the fact that $\|H\| \ge 1$.

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