ORIGINAL PAPER

ON BILATERAL GENERATING FUNCTIONS OF MODIFIED JACOBI POLYNOMIALS BY GROUP THEORETIC METHOD

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Abstract. *In this note, we have obtained some results on generating functions involving* $P_n^{(\alpha+n,\beta+n)}(x)$*, a modification of Jacobi polynomials from the group-theoretic view-point. In section 1, we have introduced a linear partial differential operator R which does not seem to have appeared in the earlier works and then we have obtained the extended form of the group generated by the operator R. Finally, in section 2, we have obtained a novel generating relation involving* $P_n^{(\alpha+n,\beta+n)}(x)$ *with the help of which we obtain some general results on bilateral generating relations of the ploynomial under consideration.*

Keywords: *Jacobi polynomials, group-theoretic method, generating function.*

1. INTRODUCTION

Generating functions play a large role in the study of special functions. Various methods have been used by researchers in the derivation of generating functions of special functions. Group theoretic method is very much potent one in comparison to other methods. From seventies and onwards of the last century, group-theoretic method has been utilized extensively by researchers in the derivation of generating functions of special functions. In the present paper, group-theoretic method has been adopted to obtain some novel results of generating functions involving $P_n^{(\alpha+n,\beta+n)}(x)$, a modification of Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$ where $P_n^{(\alpha,\beta)}(x)$ is defined by [1]:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left[\begin{matrix} -n, 1+\alpha+\beta+n; \\ & \frac{1-x}{2} \\ 1+\alpha; \end{matrix}\right] \tag{1.1}$$

In fact, while constructing the partial differential operator and the extended form of the group corresponding to the said operator for the polynomial $P_n^{(\alpha+n,\beta+n)}(x)$, we have adopted the group-theoretic method as introduced by Weisner [2].

The main results of our investigation is stated in the form of the following theorem. For previous works on bilateral generating function of Jacobi polynomials by group theoretic method one may refer to [3-10].

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Theorem: If there exists a unilateral generating function

$$G(x,\ t) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha+n,\beta+n)}(x) t^n \tag{1.2}$$

then

$$\left\{1+\frac{t}{2}(1-x)\right\}^{\alpha}\left\{1-\frac{t}{2}(1+x)\right\}^{\beta} \tag{1.3}$$

$$\times G\left(1+\frac{t}{2}(1-x^2),\ wt\left(1+\frac{t}{2}(1-x)\right)\left(1-\frac{t}{2}(1+x)\right)\right)$$

$$= \sum_{n=0}^{\infty} t^n \sigma_n(x,\ w)$$

where

$$\sigma_n(x,\ w) = \sum_{k=0}^{n} a_k \binom{n}{k} P_n^{(\alpha-n+2k,\ \beta-n+2k)}(x) w^k\ . \tag{1.4}$$

The importance of the above theorem lies in the fact that whenever one knows a unilateral generating relation of type (1.2), the corresponding bilateral generating relation can at once be written down from (1.3). Thus a large number of bilateral generating relations can be obtained by attributing different values to a_n in (1.2).

In the next section we shall first introduce a linear partial differential operator R by giving suitable interpretations to $n,\ \alpha,\ \beta$ simultaneously which does not seem to have appeared in the earlier works and then in finding the extended form of the group corresponding to R, we have obtained the following generating relation with the help of which the above theorem has been proved .

$$\left\{1+\frac{t}{2}(1+x)\right\}^{\alpha+n}\left\{1-\frac{t}{2}(1-x)\right\}^{\beta+n} \times P_n^{(\alpha+n,\ \beta+n)}\left(x-\frac{t}{2}(1-x^2)\right) \tag{1.5}$$

$$= \sum_{k=0}^{\infty} \frac{(n+1)k}{k!} P_{n+k}^{(\alpha+n-k,\ \beta+n-k)}(x) t^k\ .$$

2. DERIVATION OF OPERATOR

At first we seek a first order linear partial differential operator R :

$$R = A_1\frac{\partial}{\partial x} + A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial z} + A_4\frac{\partial}{\partial t} + A_0$$

such that

$$R\left(P_n^{(\alpha+n,\ \beta+n)}(x) y^{\alpha} z^{\beta} t^n\right) = a_n P_{n+1}^{(\alpha+n-1,\ \beta+n-1)}(x) y^{\alpha-2} z^{\beta-2} t^{n+1}, \tag{2.1}$$

where $A_i\left(i=0,\,1,\,2,\,3,\,4\right)$ are the functions of *x, y, z, t* but independent of $n,\ \alpha,\ \beta$ and a_n is a function of $n,\ \alpha,\ \beta$.

Now using the following differential recurrence relation [1]:

$$\frac{d}{dx}\left(P_n^{(\alpha,\,\beta)}\left(x\right)\right)=\frac{1}{1-x^2}\left[-2\left(n+1\right)P_n^{(\alpha-1,\,\beta-1)}\left(x\right)+\left(1+x\right)\alpha-\left(1+x\right)\beta P_n^{(\alpha,\,\beta)}\left(x\right)\right] \quad (2.2)$$

we get

$$R=\left(1-x^2\right)y^{-2}z^{-2}t\frac{\partial}{\partial x}-\left(1+x\right)y^{-1}z^{-2}t\frac{\partial}{\partial y}-\left(1+x\right)y^{-2}z^{-1}t\frac{\partial}{\partial z}-2xy^{-2}z^{-2}t^2\frac{\partial}{\partial t}$$

such that

$$R\left(P_n^{(\alpha+n,\,\beta+n)}\left(x\right)y^{\alpha}z^{\beta}t^{n}\right)=-2\left(n+1\right)P_{n+1}^{(\alpha+n-1,\,\beta+n-1)}\left(x\right)y^{\alpha-2}z^{\beta-2}t^{n+1}. \quad (2.3)$$

3. EXTENDED FORM OF THE GROUP GENERATED BY R

We now find the extended form of the group generated by R i.e., we shall find $e^{wR}f\left(x,y,z,t\right)$, where *f(x, y, z, t)* is an arbitrary function and *w* is arbitrary real or complex.

Now if $\phi(x,\,y,\,z,\,t)$ be a solution of $R\phi(x,\,y,\,z,\,t)=0$ and if we transform the differential operator R to E such that

$$E=A_1\frac{\partial}{\partial x}+A_2\frac{\partial}{\partial y}+A_3\frac{\partial}{\partial z}+A_4\frac{\partial}{\partial t}$$

then

$$E=\phi^{-1}\left(x,y,z,t\right)R\phi\left(x,y,z,t\right)$$
$$\text{i.e., } R=\phi\left(x,y,z,t\right)E\phi^{-1}\left(x,y,z,t\right).$$

Thus we get

$$\begin{aligned}e^{wR}f(x,\,y,\,z,\,t)&=e^{w\phi(x,y,z,t)E\phi^{-1}(x,y,z,t)}f(x,\,y,\,z,\,t)\\&=\phi(x,\,y,\,z,\,t)e^{wE}\left(\phi^{-1}(x,\,y,\,z,\,t)f(x,\,y,\,z,\,t)\right).\end{aligned}$$

Finally, we choose new variables X, Y, Z, T so that the operator E is transformed into the operator $D\equiv\frac{\partial}{\partial X}$. Under this change of variables, let $\phi^{-1}\left(x,y,z,t\right)f(x,\,y,\,z,\,t)$ be transformed into *F(X, Y, Z, T)*. Therefore, by Taylor's theorem, we get

$$e^{wR}f(x,\ y,\ z,\ t)=\phi(x,y,z,t)e^{wD}\left(F\left(X,\ Y,\ Z,\ T\right)\right)$$
$$=\phi(x,y,z,t)F\left(X+w,\ Y,\ Z,\ T\right)$$
$$=\phi(x,y,z,t)g(x,\ y,\ z,\ t)$$

supposing that $F\left(X+w,\ Y,\ Z,\ T\right)$ is transformed into $g(x,\ y,\ z,\ t)$ by inverse transformations.

By the method outlined above, we shall compute $e^{wR}f(x,\ y,\ z,\ t)$, where

$$R=\left(1-x^2\right)y^{-2}z^{-2}t\frac{\partial}{\partial x}-\left(1+x\right)y^{-1}z^{-2}t\frac{\partial}{\partial y}+\left(1-x\right)y^{-2}z^{-1}t\frac{\partial}{\partial z}-2xy^{-2}z^{-2}t^2\frac{\partial}{\partial t}.$$

Let $\phi(x,y,z,t)$ be a function such that $R\phi=0$.

Then on solving, we get $\phi=yzt^{-1}$.

$$\therefore \qquad E=\phi^{-1}R\phi$$
$$=\left(1-x^2\right)y^{-2}z^{-2}t\frac{\partial}{\partial x}-\left(1+x\right)y^{-1}z^{-2}t\frac{\partial}{\partial y}+\left(1-x\right)y^{-2}z^{-1}t\frac{\partial}{\partial z}-2xy^{-2}z^{-2}t^2\frac{\partial}{\partial t}.$$

Let *X, Y, Z, T* be a set of new variables for which

$$EX=1,\ EY=0,\ EZ=0,\ ET=0 \tag{3.1}$$

So that E reduces to $\frac{\partial}{\partial X}$. Solving we get

$$X=\frac{xy^2z^2}{t\left(1-x^2\right)},\ Y=\frac{1-x}{y},\ Z=\frac{1+x}{z},\ T=\frac{1-x^2}{t}$$

from which we get,

$$x=\frac{XY^2Z^2}{T},\ y=\frac{T-XY^2Z^2}{TY},\ z=\frac{T+XY^2Z^2}{TZ},\ t=\frac{T^2-X^2Y^4Z^4}{T^3}.$$

Now we are in a position to find $e^{wR}f(x,\ y,\ z,\ t)$ – the extended form of the group generated by R. Recalling that $R=\phi E\phi^{-1}$ where $\phi=yzt^{-1}$, we may write

$$e^{wR}f(x,\ y,\ z,\ t)=e^{w\phi E\phi^{-1}}f(x,\ y,\ z,\ t)$$
$$=yzt^{-1}e^{wE}\left[y^{-1}z^{-1}tf\left(x,\ y,\ z,\ t\right)\right]$$

Now the transformations

$$x=\frac{XY^2Z^2}{T},\ y=\frac{T-XY^2Z^2}{TY},\ z=\frac{T+XY^2Z^2}{TZ},\ t=\frac{T^2-X^2Y^4Z^4}{T^3}$$

will transform E into $D\left(\equiv\frac{\partial}{\partial X}\right)$. Now making this substitution and then applying Taylor's theorem, we get

$$e^{wE}\left(y^{-1}z^{-1}tf\left(x,y,z,t\right)\right)=e^{wD}\left[\frac{TY}{T-XY^2Z^2}\cdot\frac{TZ}{T+XY^2Z^2}\cdot\frac{T^2-X^2Y^4Z^4}{T^3}\right.$$
$$\left.\times f\left(\frac{XY^2Z^2}{T},\frac{TY}{T-XY^2Z^2},\frac{TZ}{T+XY^2Z^2},\frac{T^2-X^2Y^4Z^4}{T^3}\right)\right]$$
$$=e^{wD}\left[YZT^{-1}f\left(\frac{XY^2Z^2}{T},\frac{TY}{T-XY^2Z^2},\frac{TZ}{T+XY^2Z^2},\frac{T^2-X^2Y^4Z^4}{T^3}\right)\right]$$
$$=YZT^{-1}f\left[\frac{x+wy^2z^2}{T},\frac{T\left(w+X\right)y^2Z^2}{TY},\frac{T+\left(X+w\right)y^2Z^2}{TZ},\frac{T^2-\left(X+w\right)^2X^4Z^4}{T^3}\right]$$

Finally substituting

$$X=\frac{xy^2z^2}{t\left(1-x^2\right)},Y=\frac{1-x}{y},Z=\frac{1+x}{z},T=\frac{1-x^2}{t}$$

we get

$$e^{wE}\left[y^{-1}z^{-1}tf\left(x,y,z,t\right)\right]=$$
$$=y^{-1}z^{-1}tf\left(x+w\left(1-x^2\right)y^2z^2t,y\left\{1-w\left(1+x\right)y^{-2}z^{-2}t\right\},z\left\{1+w\left(1-x\right)y^{-2}z^{-2}t\right\},\right.$$
$$\left.t\left\{1-w\left(1+x\right)y^{-2}z^{-2}t\right\}\left\{1+w\left(1-x\right)y^{-2}z^{-2}t\right\}\right)$$
$$\therefore e^{wR}f\left(x,y,z,t\right)=f\left(x+w\left(1-x^2\right)y^2z^2t,y\left\{1-w\left(1+x\right)y^{-2}z^{-2}t\right\},z\left\{1+w\left(1-x\right)y^{-2}z^{-2}t\right\},\right.$$
$$\left.t\left\{1-w\left(1+x\right)y^{-2}z^{-2}t\right\}\left\{1+w\left(1-x\right)y^{-2}z^{-2}t\right\}\right)$$

4. APPLICATIONS

Now

$$e^{wR}\left(P_n^{(\alpha+n,\ \beta+n)}\left(x\right)y^{\alpha}z^{\beta}t^{n}\right)=$$
$$=P_n^{(\alpha+n,\ \beta+n)}\left(x+w\left(1-x^2\right)y^{-2}z^{-2}t\right)y^{\alpha}\left(1-w\left(1+x\right)y^{-2}z^{-2}t\right)^{\alpha}z^{\beta}$$
$$\times\left(1+w\left(1-x\right)y^{-2}z^{-2}t\right)^{\beta}t^{n}\left(1-w\left(1+x\right)y^{-2}z^{-2}t\right)^{n}\left(1+w\left(1-x\right)y^{-2}z^{-2}t\right)^{n}= \tag{4.1}$$
$$=y^{\alpha}z^{\beta}t^{n}\left\{1-w\left(1+x\right)y^{-2}z^{-2}t\right\}^{\alpha+n}\left\{1+w\left(1-x\right)y^{-2}z^{-2}t\right\}^{\beta+n}$$
$$\times P_n^{(\alpha+n,\ \beta+n)}\left(x+w\left(1-x^2\right)y^{-2}z^{-2}t\right)$$

But

$$
\begin{aligned}
& e^{wR}\left(P_n^{(\alpha+n,\ \beta+n)}(x)\, y^{\alpha} z^{\beta} t^{n}\right) = \\
& = \sum_{k=0}^{\infty} \frac{w^k}{k!} R^k \left(P_n^{(\alpha+n,\ \beta+n)}(x)\, y^{\alpha} z^{\beta} t^{n}\right) = \\
& = \sum_{k=0}^{\infty} \frac{w^k}{k!} (-2)^k (n+1)_k \left(P_{n+k}^{(\alpha+n-k,\ \beta+n-k)}(x)\, y^{\alpha-2k} z^{\beta-2k} t^{n+k}\right) = \\
& = y^{\alpha} z^{\beta} t^{n} \sum_{k=0}^{\infty} (-2w)^k \frac{(n+1)_k}{k!} \left\{P_{n+k}^{(\alpha+n-k,\ \beta+n-k)}(x) \left(y^{-2} z^{-2} t\right)^k\right\}.
\end{aligned}
\tag{4.2}
$$

Equating (4.1) and (4.2) and then putting $y^{-2}z^{-2}t = 1$, we get

$$
\begin{aligned}
& \{1 - w(1+x)\}^{\alpha+n} \{1 + w(1-x)\}^{\beta+n} P_n^{(\alpha+n,\ \beta+n)}\left(x + w(1-x^2)\right) = \\
& = \sum_{k=0}^{\infty} \frac{(n+1)_k}{k!} P_{n+k}^{(\alpha+n-k,\ \beta+n-k)}(x)(-2w)^k
\end{aligned}
\tag{4.3}
$$

Finally replacing $-2w$ by t, we get

$$
\begin{aligned}
& \left\{1 + \frac{t}{2}(1+x)\right\}^{\alpha+n} \left\{1 - \frac{t}{2}(1-x)\right\}^{\beta+n} P_n^{(\alpha+n,\ \beta+n)}\left(x - \frac{t}{2}(1-x^2)\right) = \\
& = \sum_{k=0}^{\infty} \frac{(n+1)_k}{k!} P_{n+k}^{(\alpha+n-k,\ \beta+n-k)}(x)\, t^k.
\end{aligned}
\tag{4.4}
$$

Now using the above generating relation, we shall prove the theorem.

Proof of the theorem:

Now $\sum_{n=0}^{\infty} t^n \sigma_n(x,\ w) =$

$$= \sum\nolimits_{n=0}^{\infty} \sum\nolimits_{k=0}^{\infty} t^n a_k \binom{n}{k} P_n^{(\alpha-n+2k,\ \beta-n+2k)}(x)\, w^k.$$

$$= \sum\nolimits_{n=0}^{\infty} \sum\nolimits_{k=0}^{\infty} t^{n+k} a_k \binom{n+k}{k} P_{n+k}^{(\alpha-n+k,\ \beta-n+k)}(x)\, w^k.$$

$$= \sum\nolimits_{k=0}^{\infty} a_k (wt)^k \sum\nolimits_{n=0}^{\infty} t^{n+k} a_k \binom{n+k}{k} P_n^{(\alpha-n+k,\ \beta-n+k)}(x)\, t^n.$$

$$= \sum\nolimits_{k=0}^{\infty} a_k (wt)^k \left\{1 + \frac{t}{2}(1+x)\right\}^{\alpha+k} \left\{1 - \frac{t}{2}(1-x)\right\}^{\beta+k} \times P_k^{(\alpha+k,\ \beta+k)}\left(x - \frac{t}{2}(1-x^2)\right) \quad \text{[Using (4.4)]}$$

$$=\left\{1+\frac{t}{2}(1+x)\right\}^{\alpha}\left\{1-\frac{t}{2}(1-x)\right\}^{\beta}\times\sum_{k=0}^{\infty}a_k P_k^{(\alpha+k,\ \beta+k)}\left(x-\frac{t}{2}(1-x^2)\right)\times$$
$$\times\left(tw\left\{1+\frac{t}{2}(1+x)\right\}\left\{1-\frac{t}{2}(1-x)\right\}\right)^{k}.$$
$$=\left\{1+\frac{t}{2}(1+x)\right\}^{\alpha}\left\{1-\frac{t}{2}(1-x)\right\}^{\beta}\times$$
$$\times G\left(x-\frac{t}{2}(1-x^2),tw\left(1+\frac{t}{2}(1+x)\right)1-\frac{t}{2}(1-x)\right)$$

[using(1.2)]

which is the theorem.

Finally, we would like to point it out that the theorem can be proved by the direct application of the operator *R* as follows.

Consider the formula,

$$G(x,\ t)=\sum_{n=0}^{\infty}a_n P_n^{(\alpha+n,\ \beta+n)}(x)t^n.$$

Replacing *t* by *tw* in the above formula and then multiplying both sides of the same by $y^{\alpha}z^{\beta}$, we get

$$y^{\alpha}z^{\beta}G(x,\ tw)=\sum_{n=0}^{\infty}a_n\left(P_n^{(\alpha+n,\ \beta+n)}(x)y^{\alpha}z^{\beta}t^n\right)w^n. \tag{4.5}$$

Operating *(exp wR)* on both sides of (4.5), we get

$$e^{wR}\left(y^{\alpha}z^{\beta}G(x,\ tw)\right)=e^{aR}\sum_{n=0}^{\infty}a_n\left(P_n^{(\alpha+n,\ \beta+n)}(x)y^{\alpha}z^{\beta}t^n\right)^{w^n}. \tag{4.6}$$

The left member of (4.6), with the help of (3.2) becomes

$$\begin{aligned}&y^{\alpha}z^{\beta}\left\{1-w(1+x)y^{-2}z^{-2}t\right\}^{\alpha}\left\{1+w(1-x)y^{-2}z^{-2}t\right\}^{\beta}\times\\&\times G\left(x+w\left(1-x^2\right)y^{-2}z^{-2}t,\ tw\left\{1-w(1+x)y^{-2}z^{-2}t\right\}\left\{1+w(1-x)y^{-2}z^{-2}t\right\}\right)\end{aligned} \tag{4.7}$$

The right member of (4.6) with the help of (2.3), becomes

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \frac{w^m}{m!} R^m \left(P_n^{(\alpha+n,\beta+n)}(x)\, y^\alpha z^\beta t^n\right) w^n$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \frac{w^m}{m!}(-2)^m (n+1)_m P_{n+m}^{(\alpha+n-m,\beta+n-m)}(x)\, y^{\alpha-2m} z^{\beta-2m} t^{n+m} w^n$$
$$= y^\alpha z^\beta \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n \frac{(n+1)_m}{m!}\left(-2wy^{-2}z^{-2}\right)^m t^{n+m} w^n \left(P_{n+m}^{(\alpha+n-m,\beta+n-m)}(x)\right)$$
$$= y^\alpha z^\beta \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n-m} \frac{(n-m+1)_m}{m!}\left(-2wy^{-2}z^{-2}\right)^m t^n w^{n-m} \times$$
$$\times P_n^{(\alpha+n-2m,\ \beta+n-2m)}(x) \tag{4.8}$$

Equating (4.7) and (4.8), and then replacing $\left(-2wy^{-2}z^{-2}\right)$ by *1*,we get

$$\left\{1+\frac{t}{2}(1+x)\right\}^\alpha \left\{1-\frac{t}{2}(1-x)\right\}^\beta G\left(x-\frac{t}{2}\left(1-x^2\right), tw\left\{1+\frac{t}{2}(1+x)\right\}\left\{1-\frac{t}{2}(1-x)\right\}\right)=$$
$$=\sum_{n=0}^{\infty} t^n \sum_{m=0}^{\infty} a_m \binom{n}{m} P_n^{(\alpha+2m-n,\ \beta-n+2m)}(x)\, w^m$$
$$=\sum_{n=0}^{\infty} t^n \sigma^n(x,\ w),$$

where

$$\sigma^n(x,\ w)=\sum_{m=0}^{n} a_m \binom{n}{m} P_n^{(\alpha-n+2m,\ \beta-n+2m)}(x)\, w^m,$$

which is our theorem.

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