*ORIGINAL PAPER*

# ON A PROPERTY OF THE SPLINE FUNCTIONS

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*Manuscript received: 16.07.2011; Accepted paper: 11.09.2011;*
*Published online: 01.12.2011*

***Abstract.*** *The definition of the spline functions as solutions of a variational problem is presented in the preliminaries of this paper and are shown some theorems regarding to the existence and uniqueness. The main result of this article consists of a property verified by the spline functions in connection with the spaces of functions used.*
***Keywords:*** *spline functions, variational problems, best approximation.*
***2000 Mathematics Subject Classification:*** *41A15, 41A50, 41A52.*

## 1. INTRODUCTION

**Definition 1.** Let $E_1$ be a real linear space, $(E_2, \|\cdot\|_2)$ a normed real linear space, $T: E_1 \to E_2$ an operator and $U \subseteq E_1$ a non-empty set. The problem of finding the elements $s \in U$ which satisfy

$$\|T(s)\|_2 = \inf_{u \in U} \|T(u)\|_2, \qquad (1)$$

is called the general spline interpolation problem, corresponding to the set U.

A solution of this problem, provided that exists, is named general spline interpolation element, corresponding to the set U.

The set U is called interpolatory set.

In the sequel we assume that $E_1$ is a real linear space, $(E_2, (.,.)_2, \|.\|_2)$ is a real Hilbert space, $T: E_1 \to E_2$ is a linear operator and $U \subseteq E_1$ is a non-empty convex set.

**Lemma 1.** $T(U) \subseteq E_2$ is a non-empty convex set.

The proof follows directly from the linearity of the operator $T$, taking into account that $U$ is a non-empty set.

**Theorem 1.** (Existence Theorem) If $T(U) \subseteq E_2$ is a closed set, then the general spline interpolation problem (1) (corresponding to U) has at least a solution.

The proof is shown in the papers [1, 4].

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For every element $s \in U$ we define the set

$$U(s) := U - s. \tag{2}$$

**Lemma 2.** For every element $s \in U$ the set $U(s)$ is non-empty $\left(0_{E_1} \in U(s)\right)$.

The result follows directly from the relation (2).

**Theorem 2.** (Uniqueness Theorem) If $T(U) \subseteq E_2$ is closed set and exists an element $s \in U$ solution of the general spline interpolation problem (1) (corresponding to U), such that $U(s)$ is linear subspace of $E_1$, then the following statements are true

i) For any elements $s_1, s_2 \in U$ solutions of the general spline interpolation problem (1) (corresponding to U) we have

$$s_1 - s_2 \in \operatorname{Ker}(T) \cap U(s); \tag{3}$$

ii) The element $s \in U$ is the unique solution of the general spline interpolation problem (1) (corresponding to U) if and only if

$$\operatorname{Ker}(T) \cap U(s) = \{0_{E_1}\}. \tag{4}$$

A proof is presented in the papers [2, 5].

## 2. MAIN RESULTS

**Lemma 3.** An element $s \in U$, such that $U(s)$ is linear subspace of $E_1$, is solution of the general spline interpolation problem (1) (corresponding to U) if and only if

$$(T(s), T(\tilde{u}))_2 = 0, \ (\forall)\ \tilde{u} \in U(s). \tag{5}$$

A proof is shown in the papers [2, 3].

For every element $s \in U$ we consider the set

$$S(s) := \{v \in E_1 \mid (T(v), T(\tilde{u}))_2 = 0, \ (\forall)\ \tilde{u} \in U(s)\}. \tag{6}$$

**Proposition 1.** For every element $s \in U$ the set $S(s)$ has the following properties

i) $S(s)$ is non-empty set $(0_{E_1} \in S(s))$;

ii) $S(s)$ is linear subspace of $E_1$;

iii) $\operatorname{Ker}(T) \subseteq S(s)$.

For a proof see the paper [2].

**Lemma 4.** An element $s \in U$, such that $U(s)$ is linear subspace of $E_1$, is solution of the general spline interpolation problem (1) (corresponding to U) if and only if

$$s \in S(s). \tag{7}$$

The result is a consequence of Lemma 3.

**Lemma 5.** For every element $\mathrm{s} \in \mathrm{U}$ the set $\mathrm{T}(S(\mathrm{s}))$ has the following properties

i) $\mathrm{T}(\mathrm{S}(\mathrm{s}))$ is non-empty set $(0_{\mathrm{E}_2} \in \mathrm{T}(\mathrm{S}(\mathrm{s})))$;

ii) $\mathrm{T}(\mathrm{S}(\mathrm{s}))$ is linear subspace of $\mathrm{E}_2$;

iii) $\mathrm{T}\big(\mathrm{S}(\mathrm{s})\big) \subseteq \Big(\mathrm{T}\big(\mathrm{U}(\mathrm{s})\big)\Big)^{\perp}$.

A proof is shown in the paper [1].

**Theorem 3.** If an element $\mathrm{s} \in \mathrm{U}$, such that $\mathrm{U}(\mathrm{s})$ is linear subspace of $\mathrm{E}_1$ is solution of the general spline interpolation problem (1) (corresponding to U), then the following inequality is true

$$\|\mathrm{T}(\mathrm{s}) - \mathrm{T}(\mathrm{v})\|_2 \leq \|\mathrm{T}(\mathrm{u}) - \mathrm{T}(\mathrm{v})\|_2,\ (\forall)\ \mathrm{u} \in \mathrm{U},\ (\forall)\ \mathrm{v} \in \mathrm{S}(\mathrm{s}), \tag{8}$$

with equality if and only if $\mathrm{T}(\mathrm{u}) = \mathrm{T}(\mathrm{s})$, i.e. $\mathrm{u} - \mathrm{s} \in \mathrm{Ker}(\mathrm{T})$.

*Proof.* Let $u \in U, v \in S(s)$ be arbitrary elements.

Using the properties of the inner product $(.,.)_2$ we deduce

$$\|T(u) - T(v)\|_2^2 = \|(T(u) - T(s)) + (T(s) - T(v))\|_2^2 =$$
$$= \|T(u) - T(s)\|_2^2 + 2(T(u) - T(s), T(s) - T(v))_2 + \|T(s) - T(v)\|_2^2. \tag{9}$$

As $u \in U$ and $s \in U$ it obtains $u - s \in U(s)$, therefore

$$T(u - s) \in T\big(U(s)\big). \tag{10}$$

Because $s \in U$, from Proposition 1 ii) it follows that $S(s)$ is linear subspace of $E_1$. On the other hand, as $s \in U$, such that $U(s)$ is linear subspace of $E_1$, is solution of the general spline interpolation problem (1) (corresponding to $U$), using Lemma 4 we deduce $s \in S(s)$. Also, we have $v \in S(s)$. Consequently, it follows that $s - v \in S(s)$, hence

$$T(s - v) \in T\big(S(s)\big). \tag{11}$$

Taking into account that $s \in U$ and using Lemma 5 iii), the formula (11) implies that

$$T(s - v) \in \Big(T\big(U(s)\big)\Big)^{\perp}. \tag{12}$$

From relations (10), (12) and using the property of the orthogonality we deduce

$$(T(u - s), T(s - v))_2 = 0. \tag{13}$$

As $T$ is a linear operator, the relation (13) can be written as

$$(T(u) - T(s), T(s) - T(v))_2 = 0. \tag{14}$$

Substituting the formula (14) in the equality (9) it follows that

$$\|T(u) - T(v)\|_2^2 = \|T(u) - T(s)\|_2^2 + \|T(s) - T(v)\|_2^2. \quad (15)$$

The relation (15) implies

$$\|T(s) - T(v)\|_2 \leq \|T(u) - T(v)\|_2, \quad (16)$$

with equality if and only if $\|T(u) - T(s)\|_2 = 0$, equivalent $T(u) = T(s)$, i.e. $u - s \in Ker(T)$.

**Theorem 4.** If an element $s \in U$, such that $U(s)$ is linear subspace of $E_1$ is solution of the general spline interpolation problem (1) (corresponding to U), then

$$\|T(s) - T(v)\|_2 = \inf_{u \in U} \|T(u) - T(v)\|_2, \ (\forall) v \in S(s), \quad (17)$$

i.e. $T(s)$ is the unique element in $T(U)$ of the best approximation for $T(v)$, $(\forall) v \in S(s)$.

This result follows directly from Theorem 3.

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