ORIGINAL PAPER

ON THE MIRAKJAN-FAVARD-SZÁSZ BIVARIATE APPROXIMATION FORMULA

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Abstract. *In the present paper we establish the form of remainder term associated to the Mirakjan-Favard-Szász bivariate approximation formula, using the divided differences.*

Keywords: *Mirakjan-Favard-Szász operators, univariate and bivariate approximation formula, univariate and bivariate divided differences, remainder term.*

1. INTRODUCTION

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The operators $S_n : C_2([0,+\infty)) \to C([0,+\infty))$ given by

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1}$$

where $C_2([0,+\infty)) := \left\{ f \in C([0,+\infty)) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} \textit{ exists and is finite} \right\}$ are called Mirakjan – Favard – Szász operators and were first introduced in 1941, by G. M. Mirakjan [6]. They were also intensively studied in [2] and [9].

Extension of the operators defined at (1) to the bivariate case were introduced in [8]. The operators

$$S_{m,n}(f;x,y) = e^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} f\left(\frac{i}{m}, \frac{j}{n}\right) \tag{2}$$

are called the bivariate Mirakjan-Favard-Szász operators. Extensions on the operators defined at (1) to the bivariate, as well as multivariate case were studied also in [5].

The representation of the remainder term associated to the univariate approximation formula $f(x) = S_n(f;x) + R_n(f;x)$, using the divided differences is well known and is given in [5] or [8], by

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$$R_n(f;x) = -\frac{xe^{-nx}}{n}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\left[x,\frac{k}{n},\frac{k+1}{n};f\right]. \quad (3)$$

In what follows, let $f:[0,+\infty)\times[0,+\infty)\to\mathbb{R}$ be given. The following

$$f(x,y) = S_{m,n}(f;x,y) + R_{m,n}(f;x,y) \quad (4)$$

is known as the Mirakjan – Favard – Szász bivariate approximation formula.

The aim of this paper is to revise the remainder term associated to the bivariate approximation formula (4), using bivariate divided differences. We also shall establish an upper bound estimation for the remainder term, in the case when $f\in C^{2,2}\left([0,+\infty)\times[0,+\infty)\right)$.

2. PRELIMINARIES

Let us recall some results concerning the divided differences, which we shall use in this paper. Suppose that $f:I\to\mathbb{R}$ is a given real valued function and $x_0,x_1\in I$, such that $x_0\neq x_1$, I being a certain interval of the real axis. The first order divided difference of f with respect the distinct knots x_0,x_1 is defined by

$$[x_0,x_1;f] = \frac{f(x_1)-f(x_0)}{x_1-x_0}. \quad (5)$$

If $x_0,x_1,\ldots,x_n\in I$ are distinct knots, then the n-th order divided difference of f with the mentioned knots is defined by the recurrence relation

$$[x_0,x_1,\ldots,x_n;f] = \frac{[x_1,\ldots,x_n;f]-[x_0,x_1,\ldots,x_{n-1};f]}{x_n-x_0}. \quad (6)$$

Note that the divided differences were intensively studied by T. Popoviciu [7]. Now, let $I,J\subseteq\mathbb{R}$ be real intervals, $f:I\times J\to\mathbb{R}$ be a given real valued function and $(x_0,y_0),(x_1,y_1)\in I\times J$, such that $x_0\neq x_1$ and $y_0\neq y_1$. The bivariate divided differences of f with respect the knots $(x_0,y_0),(x_1,y_1)$ are defined using the method of parametric extensions in [1], by

$$\begin{bmatrix} x_0 & x_1 \\ y_0 & y_1 \end{bmatrix}; f\Bigg] = \frac{f(x_1,y_1)-f(x_0,y_1)-f(x_1,y_0)+f(x_0,y_0)}{(x_1-x_0)(y_1-y_0)}. \quad (7)$$

Other equivalent definitions for univariate, respectively bivariate divided differences can be found in the excellent monographs [3] and [4]. In definitions of divided differences the number of abscissas in general is not equal with the number of coordinates. It follows

$$\begin{bmatrix} x_0 & x_1 \\ & y_0 \end{bmatrix}; f \Bigg] = \frac{f(x_1, y_0) - f(x_0, y_0)}{x_1 - x_0} \tag{8}$$

and

$$\left[\begin{matrix} x_0 & x_1 & x_2 \\ & y_0 & \end{matrix}; f\right] = \frac{1}{x_2 - x_1}\left(\left[\begin{matrix} x_1 & x_2 \\ & y_0 \end{matrix}; f\right] - \left[\begin{matrix} x_0 & x_1 \\ & y_0 \end{matrix}; f\right]\right), \tag{9}$$

where x_0, x_1, x_2 are distinct knots.

If $x_0, x_1, \ldots, x_p \in I$ and $y_0, y_1, \ldots, y_q \in J$ are distinct knots, the following recurrence formula

$$\begin{aligned} \left[\begin{matrix} x_0 & x_1 & \ldots & x_p \\ y_0 & y_1 & \ldots & y_q \end{matrix}; f\right] = \frac{1}{(x_p - x_0)(y_q - y_0)}\Bigg(& \left[\begin{matrix} x_1 & \ldots & x_p \\ y_1 & \ldots & y_q \end{matrix}; f\right] - \left[\begin{matrix} x_0 & \ldots & x_{p-1} \\ y_1 & \ldots & y_q \end{matrix}; f\right] \\ & - \left[\begin{matrix} x_1 & \ldots & x_p \\ y_0 & \ldots & y_{q-1} \end{matrix}; f\right] + \left[\begin{matrix} x_0 & \ldots & x_{p-1} \\ y_0 & \ldots & y_{q-1} \end{matrix}; f\right]\Bigg), \end{aligned} \tag{10}$$

holds (see [1]), for $p, q \in \mathbb{N}$, $p, q \geq 2$ and

$$\left[\begin{matrix} x_0 & x_1 & \ldots & x_p \\ y_0 & y_1 & \ldots & y_q \end{matrix}; f\right] = \left[\begin{matrix} x_{i_0} & x_{i_1} & \ldots & x_{i_p} \\ y_{j_0} & y_{j_1} & \ldots & y_{j_q} \end{matrix}; f\right], \tag{11}$$

where $(i_0, i_1, \ldots, i_p)$, $(j_0, j_1, \ldots, j_q)$ are permutation of $(0,1,\ldots,p)$, respectively $(0,1,\ldots,q)$.

3. MAIN RESULTS

Theorem 1. The remainder term associated to the bivariate approximation formula (4) can be represented under the following form

$$\begin{aligned} R_{m,n}(f; x, y) = & -\frac{xe^{-(mx+ny)}}{m} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left[\begin{matrix} x & \frac{i}{m} & \frac{i+1}{m} \\ & \frac{j}{n} & \end{matrix}; f\right] \\ & - \frac{ye^{-(mx+ny)}}{n} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left[\begin{matrix} & \frac{i}{m} & \\ y & \frac{j}{n} & \frac{j+1}{n} \end{matrix}; f\right] \\ & + \frac{xye^{-(mx+ny)}}{mn} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left[\begin{matrix} x & \frac{i}{m} & \frac{i+1}{m} \\ y & \frac{j}{n} & \frac{j+1}{n} \end{matrix}; f\right], \end{aligned} \tag{12}$$

for any $(x,y)\in[0,+\infty)\times[0,+\infty)$ and any $m,n\in\mathbb{N}$.

Proof. In order to evaluate the remainder term, we first notice that the bivariate Mirakjan-Favard-Szász operators reproduce constants, such that from (4), one arrives at

$$R_{m,n}(f;x,y)=e^{-(mx+ny)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\left(f(x,y)-f\left(\frac{i}{m},\frac{j}{n}\right)\right). \tag{13}$$

Using the identity

$$f(x,y)-f\left(\frac{i}{m},\frac{j}{n}\right)=\left(f\left(x,\frac{j}{n}\right)-f\left(\frac{i}{m},\frac{j}{n}\right)\right)+\left(f\left(\frac{i}{m},y\right)-f\left(\frac{i}{m},\frac{j}{n}\right)\right)$$
$$+\left(f(x,y)-f\left(x,\frac{j}{n}\right)-f\left(\frac{i}{m},y\right)+f\left(\frac{i}{m},\frac{j}{n}\right)\right)$$

and taking (13) into account, it follows

$$R_{m,n}(f;x,y)=e^{-(mx+ny)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\left(f\left(x,\frac{j}{n}\right)-f\left(\frac{i}{m},\frac{j}{n}\right)\right)$$
$$+e^{-(mx+ny)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\left(f\left(\frac{i}{m},y\right)-f\left(\frac{i}{m},\frac{j}{n}\right)\right)$$
$$+e^{-(mx+ny)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\left(f(x,y)-f\left(x,\frac{j}{n}\right)-f\left(\frac{i}{m},y\right)+f\left(\frac{i}{m},\frac{j}{n}\right)\right).$$

Denoting

$$S_1=e^{-(mx+ny)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\left(f\left(x,\frac{j}{n}\right)-f\left(\frac{i}{m},\frac{j}{n}\right)\right),$$
$$S_2=e^{-(mx+ny)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\left(f\left(\frac{i}{m},y\right)-f\left(\frac{i}{m},\frac{j}{n}\right)\right),$$
$$S_3=e^{-(mx+ny)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\left(f(x,y)-f\left(x,\frac{j}{n}\right)-f\left(\frac{i}{m},y\right)+f\left(\frac{i}{m},\frac{j}{n}\right)\right),$$

we get $R_{m,n}(f;x,y)=S_1+S_2+S_3$. For S_1, taking (8) and (9) into account, it follows

$$S_1 = e^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left(x - \frac{i}{m} \right) \begin{bmatrix} x & \frac{i}{m} \\ \frac{j}{n} & \end{bmatrix}; f$$

$$= xe^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \begin{bmatrix} x & \frac{i}{m} \\ \frac{j}{n} & \end{bmatrix}; f - xe^{-(mx+ny)} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^{i-1}}{(i-1)!} \frac{(ny)^j}{j!} \begin{bmatrix} x & \frac{i}{m} \\ \frac{j}{n} & \end{bmatrix}; f$$

$$= xe^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \begin{bmatrix} x & \frac{i}{m} \\ \frac{j}{n} & \end{bmatrix}; f - xe^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \begin{bmatrix} x & \frac{i+1}{m} \\ \frac{j}{n} & \end{bmatrix}; f$$

$$= xe^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left(\begin{bmatrix} x & \frac{i}{m} \\ \frac{j}{n} & \end{bmatrix}; f - \begin{bmatrix} x & \frac{i+1}{m} \\ \frac{j}{n} & \end{bmatrix}; f \right)$$

$$= -\frac{xe^{-(mx+ny)}}{m} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \begin{bmatrix} x & \frac{i}{m} & \frac{i+1}{m} \\ & \frac{j}{n} & \end{bmatrix}; f.$$

Analogous for S_2, it follows

$$S_2 = e^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left(y - \frac{j}{n} \right) \begin{bmatrix} & \frac{i}{m} \\ y & \frac{j}{n} \end{bmatrix}; f$$

$$= \ldots = -\frac{ye^{-(mx+ny)}}{n} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \begin{bmatrix} & \frac{i}{m} & \\ y & \frac{j}{n} & \frac{j+1}{n} \end{bmatrix}; f.$$

For S_3, taking (10) and (11) into account, it follows

$$S_3 = e^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left(x - \frac{i}{m} \right) \left(y - \frac{j}{n} \right) \left[\begin{matrix} x & \frac{i}{m} \\ y & \frac{j}{n} \end{matrix} ; f \right]$$

$$= xye^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left[\begin{matrix} x & \frac{i}{m} \\ y & \frac{j}{n} \end{matrix} ; f \right] - xye^{-(mx+ny)} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^{i-1}}{(i-1)!} \frac{(ny)^j}{j!} \left[\begin{matrix} x & \frac{i}{m} \\ y & \frac{j}{n} \end{matrix} ; f \right]$$

$$- xye^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^{j-1}}{(j-1)!} \left[\begin{matrix} x & \frac{i}{m} \\ y & \frac{j}{n} \end{matrix} ; f \right] + xye^{-(mx+ny)} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{(mx)^{i-1}}{(i-1)!} \frac{(ny)^{j-1}}{(j-1)!} \left[\begin{matrix} x & \frac{i}{m} \\ y & \frac{j}{n} \end{matrix} ; f \right]$$

$$= xye^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left[\begin{matrix} x & \frac{i}{m} \\ y & \frac{j}{n} \end{matrix} ; f \right] - xye^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left[\begin{matrix} x & \frac{i+1}{m} \\ y & \frac{j}{n} \end{matrix} ; f \right]$$

$$- xye^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left[\begin{matrix} x & \frac{i}{m} \\ y & \frac{j+1}{n} \end{matrix} ; f \right] + xye^{-(mx+ny)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left[\begin{matrix} x & \frac{i+1}{m} \\ y & \frac{j+1}{n} \end{matrix} ; f \right]$$

$$= \frac{xye^{-(mx+ny)}}{mn} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(mx)^i}{i!} \frac{(ny)^j}{j!} \left[\begin{matrix} x & \frac{i}{m} & \frac{i+1}{m} \\ y & \frac{j}{n} & \frac{j+1}{n} \end{matrix} ; f \right].$$

The upper bound estimation for the remainder term is given in:

Theorem 2. If the function f has the following properties

- $f \in C^{2,2}\left([0,+\infty) \times [0,+\infty)\right)$,
- there exists $\dfrac{\partial^4 f}{\partial x^2 \partial y^2}$ on $(0,+\infty) \times (0,+\infty)$,
- $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^4 f}{\partial x^2 \partial y^2}$ are bounded on $(0,+\infty) \times (0,+\infty)$, then the inequality

$$\left| R_{m,n}(f; x, y) \right| \le \frac{x}{2m} M_{2,0}[f] + \frac{y}{2n} M_{0,2}[f] + \frac{xy}{4mn} M_{2,2}[f] \tag{14}$$

holds, for any $(x, y) \in [0,+\infty) \times [0,+\infty)$ and any $m, n \in \mathbb{N}$, where

$$M_{2,0}[f] := \sup_{(x,y) \in (0,+\infty) \times (0,+\infty)} \left| \frac{\partial^2 f}{\partial x^2}(x, y) \right|, \quad M_{0,2}[f] := \sup_{(x,y) \in (0,+\infty) \times (0,+\infty)} \left| \frac{\partial^2 f}{\partial y^2}(x, y) \right|,$$

$$M_{2,2}[f] := \sup_{(x,y)\in(0,+\infty)\times(0,+\infty)} \left| \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \right|.$$

Proof. Applying the mean value theorem for the bivariate divided differences, it follows that exist $(\xi_1(i,j),\eta_1(i,j)),(\xi_2(i,j),\eta_2(i,j)),(\xi_3(i,j),\eta_3(i,j)) \in (0,+\infty)\times(0,+\infty)$, such that

$$\begin{aligned} R_{m,n}(f;x,y) = &-\frac{xe^{-(mx+ny)}}{m}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\xi_1(i,j),\eta_1(i,j)) \\ &-\frac{ye^{-(mx+ny)}}{n}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\xi_2(i,j),\eta_2(i,j)) \\ &+\frac{xye^{-(mx+ny)}}{mn}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}\frac{1}{4}\frac{\partial^4 f}{\partial x^2\partial y^2}(\xi_3(i,j),\eta_3(i,j)). \end{aligned}$$

By using modulus, the fact that partial derivatives of function f are bounded on $(0,+\infty)\times(0,+\infty)$ and $e^{-(mx+ny)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(mx)^i}{i!}\frac{(ny)^j}{j!}=1$, one arrives at (14).

Remark 3. The results presented above differ from the results established by F. Stancu and C. Manole for the representation of remainder term associated to the bivariate approximation formula of Mirakjan-Favard-Szász operators.

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