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ORIGINAL PAPER

VARIOUS SPACES FOR BANACH SPACE VALUED POTENTIAL HANKEL TRANSFORM

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Abstract: *Different testing function spaces are defined for Banach space valued Potential –Hankel transform by using Gelfand-Shilov technique of S_α spaces, some properties of the spaces are also proved. At the end generalized Banach space valued Potential – Hankel transform is defined.*

Keywords: *Banach Space, Potential Transform, Hankel Transform.*

1. INTRODUCTION

In realizability theory for electric systems , many systems consist of signals having instantaneous values in Banach space. Zemanian had focused on Banach space valued Laplace transform in [4]. Following [4], Tekale studied Banach space valued Stieltijes transform in [2]. We have discussed Potential transform of Banach space valued in [3].

In this paper, we have introduced Banach space valued Potential – Hankel transform. For which we have defined various spaces of Gelfand – Shilov type.

Gelfand – Shilov in [1] had given spaces of type S_α in which conditions are imposed not only on the decrease of the function but also on the growth of their derivatives as the order of derivative increases.

In this paper the spaces $PH_\alpha(A)$, $\mathrm{PH}^\beta(\mathrm{A})$, $\mathrm{PH}_\alpha^\beta(\mathrm{A})$ etc are defined and their properties are discussed. Also generalized Potential – Hankel transformation is introduced. Notation and terminology as per Zemanian A.H.[4] and [5].

2. GELFAND-SHILOV TYPE SPACE

2.1 VARIOUS SPACES:

The space $\mathrm{PH}_\alpha(\mathrm{A})$: For $\alpha \geq 0$,

$$P_{c,d}H_\alpha(A) = PH_\alpha(A) =$$

$$= \left\{ \phi : \phi \in E_+, \gamma_{c,d,k,l,q}\phi(t,s)\} = \sup_{I_1} \left\| \lambda_{c,d}(t) s^1 (tD_t)^k (s^{-1}D)^q \left[ts^{-\mu-\frac{1}{2}} \phi(t,s) \right] \right\|_A \leq C_{k,q} A_1^l, l^{l\alpha} \right\}$$

for all $k, q \in IN_0$, where the constants A_1 & C_{kq} depend on the function ϕ.

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The space $\mathrm{PH}^{\beta}(\mathrm{A})$: For $\beta \geq 0$,

$$PH^{\beta}(A)=\left\{\phi:\phi\in E_{+},\gamma_{r,l,q}\phi(t,s)=\sup_{I_1}\left\|\lambda_{c,d}(t)s^{l}(tD_{t})^{k}(s^{-1}D)^{q}\left[ts^{-\mu-\frac{1}{2}}\phi(t,s)\right]\right\|_{A}\leq C_{k,l}A_{2}^{q}q^{q\beta}\right\}$$

The space $PH_{\alpha}^{\beta}(A)$: For $\alpha\geq 0$ and $\beta \geq 0$,

$$PH_{\alpha}^{\beta}(A)=\left\{\phi:\phi\in E_{+},\gamma_{c,d,k,l,q}\phi(t,s)=\sup_{I_1}\left\|\lambda_{c,d}(t)s^{l}(tD_{t})^{k}(s^{-1}D)^{q}\left[ts^{-\mu-\frac{1}{2}}\phi(t,s)\right]\right\|_{A}\leq C_{k}A_{1}^{l}A_{2}^{q}l^{l\alpha}q^{q\beta}\right\}$$

The space $P_{\gamma}H(A)$: For $\gamma\geq 0$,

$$P_{\gamma}H(A)=\{\phi:\phi\in E_{+};\gamma_{c,d,k,l,q}\phi(t,s)=\sup_{I_1}\left\|\lambda_{c,d}(t)s^{l}(tD_{t})^{k}(s^{-1}D)^{q}\left[ts^{-\mu-\frac{1}{2}}\phi(t,s)\right]\right\|_{A}\leq \mathrm{C}_{l,q}A_{4}^{k}k^{k\gamma}\}$$

The space $PH_{\alpha,m}(A)$: For given $m>0$.

The space it set as,

$$PH_{\alpha,m}=\{\phi:\phi\in E_{+};\gamma_{k,l,q}\phi(t,s)=\sup_{I_1}\left\|\lambda_{c,d}(t)s^{l}(tD_{t})^{k}(s^{-1}D)^{q}\left[ts^{-\mu-\frac{1}{2}}\phi(t,s)\right]\right\|_{A} \qquad (2.1.1)$$

for any δ>0. $\leq \mathrm{C}_{k,q,\delta}(m+\delta)^{l}l^{l\alpha}\}$,

The space $PH^{\beta,n}(A)$: For given n>0,

The space is defined as,

$$PH^{\beta,n}=\{\phi:\phi\in E_{+};\gamma_{k,l,q}\phi(t,s)=\sup_{I_1}\left\|\lambda_{c,d}(t)s^{l}(tD_{t})^{k}(s^{-1}D)^{q}\left[ts^{-\mu-\frac{1}{2}}\phi(t,s)\right]\right\|_{A} \qquad (2.1.2)$$

for any η>0. $\leq \mathrm{C}_{k,l,n}(n+\eta)^{q}q^{q\beta}\}$,

The space $PH_{\alpha,m}^{\beta,n}(A)$:

This space is formed by combining the conditions of (2.1.1) and (2.1.2) .

$$PH_{\alpha,m}^{\beta,n}=\{\phi:\phi\in E_{+};\gamma_{k,l,q}\phi(t,s)=\sup_{I_1}\left\|\lambda_{c,d}(t)s^{l}(tD_{t})^{k}(s^{-1}D)^{q}\left[ts^{-\mu-\frac{1}{2}}\phi(t,s)\right]\right\|_{A}$$
$$\leq \mathrm{C}_{k}(m+\delta)^{l}(n+\eta)^{q},l^{l\alpha}q^{q\beta}\},$$

for given $m,n>0$ and for any $\delta,n>0$.

The space $P_{\gamma,\Delta}H(A)$:

For given $\Delta>0$, the space is defined as,

$$P_{\gamma,\Delta}H(A)=\{\phi:\phi\in E_{+};\gamma_{k,l,q}\phi(t,s)=\sup_{I_1}\left\|\lambda_{c,d}(t)s^{l}(tD_{t})^{k}(s^{-1}D)^{q}\left[ts^{-\mu-\frac{1}{2}}\phi(t,s)\right]\right\|_{A}$$
$$\leq \mathrm{C}_{l,q,\zeta}(\Delta+\zeta)^{k}k^{k\gamma}\},$$

2.2. THEOREM:

$PH_\alpha(A)$ is a Frechet space.

Proof: As the family D_α of seminorms $\{\gamma_{c,d,kq}\}_{k,q}^{\infty}$ generating T_α is countable, it suffices to prove the completeness of the space PH_α.

Let us consider a Cauchy sequence $\{\psi_n\}$ in $PH_\alpha(A)$. Hence, for a given $\in > 0$, there exist an $N = N_{kq}$ such that for $a, b \geq N$,

$$\gamma_{c,d,k,l,q}(\phi_a - \phi_b) = \sup_{I_1} \left\| \lambda_{c,d}(t) s^l (tD_t)^k (s^{-1}D)^q \left[ts^{-\mu-\frac{1}{2}} \phi(\phi_a - \phi_b) \right] \right\|_A < \in .$$

In particular for $k, q = 0$, for $a, b \geq N$.

$$\sup \left\| \lambda_{c,d}(t) s^l (tD_t)^k (s^{-1}D)^q \left[ts^{-\mu-\frac{1}{2}} \phi_a(t,s) \right] \right\|_A < \in .$$

Consequently for fixed *(t,s)* in I_1 where I_1 is open set $(R_+)\times(R_+)$ [first quandrant] $\{\phi_a(t,s)\}$ is a numerical Cauchy sequence. Let $\phi(t,s)$ be the pointwise limit of $\{\phi_a(t,s)\}$.

Using equation (2.2.2) it can be easily deduced that $\{\phi_a\}$ converges to ϕ uniformly on $I_{1.}$

Thus ϕ is continuous, moreover repeated use of equation (2.2.1) for different value of *k,q* and the use of the proposition from topological linear spaces and duality yields that ϕ is smooth i.e. $\phi \in E_+$.

Further from ecuation (2.2.1) we get,

$$\gamma_{c,d,k,l,q(\phi_a)} \leq \gamma_{c,d,k,l,q(\phi_b)} + \in \leq C_{k,q} A_1^l l^{l\alpha} + \in$$

for all $a \geq N$.

Letting $a \to \infty$ and observing that $\in$ is arbitrary we get

$$\gamma_{c,d,k,l,q} = \sup \left\| \lambda_{c,d}(t) s^l (tD_t)^k (s^{-1}D)^q \left[ts^{-\mu-\frac{1}{2}} \phi(\phi_a - \phi_b) \right] \right\|_A \leq C_{k,q} A_1^l l^{l\alpha}.$$

Hence $\psi \in PH_\alpha(A)$ and it is the T_α limit of ψ_n by equation (2.2.1) again. This proves the completeness of $PH_\alpha(A)$, and our proof is complete.

The non triviality of these spaces is proved in the following theorem.

2.3. THEOREM

The space D(I_1) is a subspace of $PH_\alpha(A)$ such that the injection mapping from D(I_1) to PH_α is continuous i.e. $T / D(I_1) \subset T(I_1)$

Proof: For $\phi(t,s) \in D(I_1)$, set

$$L = \sup\{ s : (t,s)\text{supp } \phi \}$$

and

$$C_{k,q} = \sup \left\| \lambda_{c,d}(t) s^1 (tD_t)^k (s^{-1}D)^q \left[ts^{-\mu-\frac{1}{2}} \phi(t,s) \right] \right\|_A .$$

Then

$$C_{k,q} = \sup \left\| \lambda_{c,d}(t) s^1 (tD_t)^k (s^{-1}D)^q \left[ts^{-\mu-\frac{1}{2}} \phi(t,s) \right] \right\|_A \le C_{k,q} L^1 \le C_{k,q} \left(\frac{L}{A_1 l^{\alpha}} \right)^1 A_1^l l^{l\alpha}.$$

Since $\left(\frac{L}{A_1 l^{\alpha}} \right) \le 1$ if $l \ge \left(\frac{L}{A_1} \right)^{\frac{1}{\alpha}}$ define $l_0 \ge \left[\left(\frac{L}{A_1} \right)^{\frac{1}{\alpha}} \right] + 1$ where [X] denotes the Gaussian symbol that is the greatest integer not exceeding X.

Therefore for $l > l_0$ we have,

$$\gamma_{c,d,k,l,q} \phi(t,s) \le C_{k,q} A_1^l l^{l\alpha} \tag{2.3.1}$$

If $k \le k_0$, let us write

$$C = \max \left\{ \left(\frac{L}{A_1} \right), \left(\frac{L}{A_1 2^{\alpha}} \right)^2, \left(\frac{L}{A_1 3^{\alpha}} \right)^3, \ldots\ldots \left(\frac{L}{A_1 l_0^{\alpha}} \right)^1 \right\}$$

Then again from equation (2.3.1)

$$\gamma_{c,d,k,l,q} \phi(t,s) \le C \cdot C_{k,q} A_1^l l^{l\alpha} \tag{2.3.2}$$

Hence the inequalities (2.3.1) and (2.3.2) together yields

$$\gamma_{c,d,k,l,q} \phi(t,s) \le C'_{k,q} A_1^l l^{l\alpha}$$

implying that $\phi \in PH_{\alpha}(A)$

To prove the continuity of the injection map, consider a sequence $\{\phi_n\}$ in $D(I_1)$ converging to zero then,

$$\gamma_{c,d,k,l,q} \phi_b(t,s) = \sup \left\| \lambda_{c,d}(t) s^l (tD_t)^k (s^{-1}D)^q \left[ts^{-\mu-\frac{1}{2}} \phi_b(t,s) \right] \right\|_A . \tag{2.3.3}$$

Now from the inductive limit, the subset K_m (say) of I_1 such that $\phi_b D(k_m)$ for each $b \ge 1$ and $\phi_b \to 0$ in $D(k_m)$.

If $M = \sup_{k_m} \left\| \lambda_{c,d}(t) s^l s^q \right\|$

Then form equation (2.3.3)

$$\gamma_{c,d,k,l,q} \phi_b(t,s) \le M \sup_{k_m} \left\| D^{k+1} \phi_b(t,s) \right\|_A \to 0$$

as $b \to \infty$

Hence $\phi_b \to 0$ in $PH_{\alpha}(A)$ and therefore

$$T_{\alpha} / D(I_1) \subset T(I_1)$$

Analogously, we can prove similar results for other spaces defined.

2.4. PROPERTIES OF THE SPACES $PH_\alpha\left(A\right)$ *AND* $PH_\alpha^\beta\left(A\right)$:

Proposition: If $\alpha_1 < \alpha_2$ & $\beta_1 < \beta_2$ then $PH_{\alpha_1}^{\beta_1}\left(A\right) \subset PH_{\alpha_2}^{\beta_2}\left(A\right)$ and the topology of $PH_{\alpha_1}^{\beta_1}\left(A\right)$ is equivalent to the topology induced on the topology $PH_{\alpha_1}^{\beta_1}\left(A\right)$ by $PH_{\alpha_2}^{\beta_2}\left(A\right)$.

Proof: Let $\phi \in PH_{\alpha_1}^{\beta_1}(A)$

$$\gamma_{c,d,k,l,q}\phi\left(t,s\right) = \sup_{I_1}\left\|\lambda_{c,d}(t)s^l(tD_t)^k(s^{-1}D)^q\left[ts^{-\mu-\frac{1}{2}}\phi(t,s)\right]\right\|_A \le C_k A_1^l A_2^q l^{l\alpha_1} q^{q\beta_1} \le C_k A_1^l A_2^q l^{l\alpha_2} q^{q\beta_2}$$

Hence $\phi \in PH_{\alpha_2}^{\beta_2}(A)$

Consequently $PH_{\alpha_1}^{\beta_1}(A) \subset PH_{\alpha_2}^{\beta_2}(A)$.

The topology of $PH_{\alpha_1}^{\beta_1}(A)$ is equivalent to the topology $T_{\alpha_2}^{\beta_2} / PH_{\alpha_1}^{\beta_1}(A)$. It is clear from the definition of topologies of these spaces.

Proposition: If $m_1 < m_2$ then $PH_{\alpha,m_1}(A) \subset PH_{\alpha,m_2}(A)$.

The topology of $PH_{\alpha,m_1}(A)$ is equivalent to the topology induced on $PH_{\alpha,m_1}(A)$ by $PH_{\alpha,m_2}(A)$ i.e. $T_{\alpha,m}T_{\alpha,m_2} / PH_{\alpha,m_1}$

Proof: Let $\phi \in PH_{\alpha_1 m_1}\left(A\right) \therefore \gamma_{k,l,q}\phi\left(t,s\right) \le C_{k,q,\delta}\left(m_1+\delta\right)^1 l^{l\alpha} \le C_{k,q,\delta}\left(m_1+\delta\right)^1 l^{l\alpha}$

Thus $PH_{\alpha_1 m_1}\left(A\right) \subset PH_{\alpha,m_2}\left(A\right)$.

The second part is clear from the definition of topologies of these spaces.

We now define inductive limit of the spaces $PH_{\alpha,m}\left(A\right): m \ge 1$ as follows:

2.5. THEOREM:

$PH_\alpha(A) = \bigcup_m PH_{\alpha,m}(A)$ and if the space $PH_\alpha(A)$ is equipped with the strict inductive limit topology $T_{\alpha,m}$ defined by the injection map from $PH_{\alpha,m}\left(A\right)$ to $PH_\alpha(A)$, then the sequence $\{\phi_n\}$ in $PH_\alpha(A)$ converges to zero if $\{\phi_n\}$ is contained in some $PH_{\alpha,m}(A)$ and converges there into zero.

Proof: The above theorem is immediate consequence of theorem from topological spaces and duality.

Once we show that

$$PH_\alpha(A) = \bigcup_m PH_{\alpha,m}(A).$$

Clearly,

$$\bigcup_{m-1}^{\infty} PH_{\alpha,m}(A) \subset PH_\alpha(A).$$

For proving another inclusion, let $\phi \in PH_\alpha\left(A\right)$ then

$$\gamma_{c,d,k,l,q}\phi(t,s) = \sup_{I_1}\left\|\lambda_{c,d}(t)s^l(tD_t)^k(s^{-1}D)^q\left[ts^{-\mu-\frac{1}{2}}\phi(t,s)\right]\right\|_A \le C_{k,q}A_1^l l^{l\alpha}, \qquad (2.5.1)$$

where A_1 is some positive constant.

Choose an integer depending on the value of A_1 and $\delta > 0$ such that

$$C_{k,q}A_1^l \leq C_{k,q}(m+\delta)^l.$$

Then from equation (2.5.1), we immediately get $\phi \in PH_\alpha(A)$ implying that

$$PH_\alpha(A) \subset \bigcup_{m-1}^{\infty} PH_{\alpha,m}(A)$$

Remark: Analogous to above theorem we can show that the spaces $PH^\beta(A), PH_\alpha^\beta(A), PH_y(A)$ are the inductive limits of the corresponding spaces $PH^{\beta,n}(A), PH_{\alpha,m}^{\beta,n}(A), PH_{y,\Delta}(A)$.

3. GENERALISED POTENTIAL HANKEL TRANSFORMS ON $PH_\alpha(A)$ SPACES.

Let $f \in (D_+;A)$.

f is said to be Potential Hankel transformable, if there exist $\sigma_1, \sigma_2, \sigma_1, \sigma_2 \in [-\infty, \infty]$ such that $\sigma_1 < \sigma_2, \sigma_1^{'} < \sigma_2^{'}$, $f \in [PH_\alpha(\sigma_1,\sigma_2);A]$ and in addition $f \notin [PH_\alpha(\omega,z);A]$ and if either $\omega < \sigma_1$ or $z < \sigma_2$, $\omega < \sigma_1^{'}$ or $z < \sigma_2^{'}$

$$\Omega_f = \left\{(y,x) : \sigma_1 < \operatorname{Re} y < \sigma_2 \text{ and } \sigma_1^{'} < \operatorname{Re} x < \sigma_2^{'}\right\}$$

with $\dfrac{t}{y^2+t^2} \in PH_\alpha(\sigma_1,\sigma_2), \sqrt{xs}\,J\mu(x,s) \in PH_\alpha(\sigma_1^{'},\sigma_2^{'})$.

Then the Potential Hankel transform is defined as ,

$$PH\left\{f(t,s)\right\} = F(y,x) = \left\langle f(t,s)\frac{t}{y^2+t^2}\sqrt{xs}\,J\mu(x,s)\right\rangle,$$

for $(y,x) \in \Omega_f$

$F(y,x)$ is an A-valued analytic function on Ω_f.

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