ORIGINAL PAPER

SOME SEQUENCES THAT CONVERGE TO A GENERALIZATION OF EULER'S CONSTANT

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Abstract. *We consider a generalization of Euler's constant as the limit* $\gamma(a)$ *of the sequence*

$$\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n+1}-\ln\frac{a+n-1}{a}\right)_{n\in N},$$

where $a\in(0,+\infty)$. *The purpose of this paper is to give some sequences that converge to* $\gamma(a)$.

Keywords: *sequence, convergence, Euler's constant.*
Mathematics Subject Classification: *11Y60, 40A05.*

1. INTRODUCTION

Euler's constant, usually denoted by γ, is the limit of the sequence $(D_n)_{n\in N}$ defined by $D_n=1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n$, for each $n\in N$. It is well-known that $\lim_{n\to\infty} n(D_n-\gamma)=\frac{1}{2}$ (see [1 – 3, 5, 7, 13, 14, 22, 24-27]. This means that the sequence $(D_n)_{n\in N}$ converges slowly to $\gamma=0.5772156649\ldots$, more precisely, with order 1.

Sequences that converge faster to γ were given in the literature. D. W. DeTemple proved in [4] that $\frac{1}{24(n+1)^2}<R_n-\gamma<\frac{1}{24n^2}$, for each $n\in N$, where

$$R_n=1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln\left(n+\frac{1}{2}\right),$$

for each $n\in N$. So, the sequence $(R_n)_{n\in N}$ converges to γ with order 2.

Considering a sequence used by L. Tóth in [23], namely the sequence $(T_n)_{n\in N}$ defined by

$$T_n=1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln\left(n+\frac{1}{2}+\frac{1}{24n}\right),$$

for each $n\in N$, T. Negoi proved in [12] that $\frac{1}{48(n+1)^3}<\gamma-T_n<\frac{1}{48n^3}$, for each $n\in N$. As can be seen, the sequence $(T_n)_{n\in N}$ converges to γ with order 3.

Let $a\in(0,+\infty)$. We consider the sequence $(y_n(a))_{n\in N}$ defined by

$$y_n(a)=\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a},$$

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for each $n \in N$. The sequence $(y_n(a))_{n \in N}$ is convergent (see, for example, [6, p. 453]; see also [15 - 20] and some of the references therein) and its limit, denoted by $\gamma(a)$, is a generalization of Euler's constant. We have $\gamma(1) = \gamma$.

Results regarding $\gamma(a)$ we have obtained in [15 - 21].

In Section 2 we give sequences that converge to $\gamma(a)$, some of them with order 4.

We remind the following lemma (C. Mortici [8, Lemma]), which is a consequence of the Stolz-Cesaro Theorem, the $\frac{0}{0}$ case. Applications of this lemma in obtaining sequences that converge to $\gamma(a)$ or γ can be found, for example, in [9 - 11].

Lemma 1.1. *Let $(x_n)_{n \in N}$ be a convergent sequence of real numbers and $x^* = \lim_{n \to \infty} x_n$. We suppose that there exists $\alpha \in R$, $\alpha > 1$, such that*

$$\lim_{n \to \infty} n^{\alpha}(x_n - x_{n+1}) = l \in \overline{R}.$$

Then there exists the limit

$$\lim_{n \to \infty} n^{\alpha - 1}(x_n - x^*) = \frac{l}{\alpha - 1}.$$

2. SEQUENCES THAT CONVERGE TO $\gamma(a)$

Theorem 2.1. *Let $a \in (0, +\infty)$ and $b, c, d \in R$. Let $n_0 \in N$ be such that $a + n - 1 + c + \frac{d}{a+n-1} > 0$, for each $n \in N$, with $n \geq n_0$. We consider the sequence $(v_n(a,b,c,d))_{n \geq n_0}$ defined by*

$$v_n(a,b,c,d) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} + \frac{b}{a+n-1} - \ln\left(\frac{a+n-1}{a} + \frac{c}{a} + \frac{d}{a(a+n-1)}\right),$$

for each $n \in N$, with $n \geq n_0$. Also, we specify that $\gamma(a)$ is the limit of the sequence $(y_n(a))_{n \in N}$ from Introduction.

(i) If $b \neq c - \frac{1}{2}$, then

$$\lim_{n \to \infty} n\big(v_n(a,b,c,d) - \gamma(a)\big) = b - c + \frac{1}{2}.$$

(ii) If $b = c - \frac{1}{2}$ and $d \neq \frac{1}{2}\left(c^2 - \frac{1}{6}\right)$, then

$$\lim_{n \to \infty} n^2\left(v_n\left(a, c - \frac{1}{2}, c, d\right) - \gamma(a)\right) = \frac{1}{2}\left(c^2 - \frac{1}{6}\right) - d.$$

(iii) If $b = c - \frac{1}{2}$, $d = \frac{1}{2}\left(c^2 - \frac{1}{6}\right)$ and $c \neq 0$, $c \neq \pm\frac{\sqrt{2}}{2}$, then

$$\lim_{n \to \infty} n^3\left(v_n\left(a, c - \frac{1}{2}, c, \frac{1}{2}\left(c^2 - \frac{1}{6}\right)\right) - \gamma(a)\right) = \frac{c}{6}\left(c^2 - \frac{1}{2}\right).$$

(iv) If $b = c - \frac{1}{2}$, $d = \frac{1}{2}\left(c^2 - \frac{1}{6}\right)$ and $c = 0$, then

$$\lim_{n \to \infty} n^4\left(v_n\left(a, -\frac{1}{2}, 0, -\frac{1}{12}\right) - \gamma(a)\right) = \frac{17}{1440}.$$

(v) If $b = c - \frac{1}{2}$, $d = \frac{1}{2}\left(c^2 - \frac{1}{6}\right)$ and $c = \frac{\sqrt{2}}{2}$, then

$$\lim_{n \to \infty} n^4\left(v_n\left(a, \frac{\sqrt{2}-1}{2}, \frac{\sqrt{2}}{2}, \frac{1}{6}\right) - \gamma(a)\right) = \frac{1}{720}.$$

(vi) If $b = c - \frac{1}{2}$, $d = \frac{1}{2}\left(c^2 - \frac{1}{6}\right)$ and $c = -\frac{\sqrt{2}}{2}$, then

$$\lim_{n\to\infty} n^4\left(v_n\left(a,-\frac{\sqrt{2}+1}{2},-\frac{\sqrt{2}}{2},\frac{1}{6}\right)-\gamma(a)\right)=\frac{1}{720}.$$

Proof. Clearly, $\lim_{n\to\infty} v_n(a,b,c,d)=\gamma(a)$. We have

$$\begin{aligned}
&v_n(a,b,c,d)-v_{n+1}(a,b,c,d)\\
&=\frac{b}{a+n-1}-\frac{b+1}{a+n}-\ln\left(a+n-1+c+\frac{d}{a+n-1}\right)+\ln\left(a+n+c+\frac{d}{a+n}\right)\\
&=\frac{b}{(a+n)\left(1-\frac{1}{a+n}\right)}-\frac{b+1}{a+n}-\ln\left(1+\frac{c-1}{a+n}+\frac{d}{(a+n)^2\left(1-\frac{1}{a+n}\right)}\right)\\
&\quad+\ln\left(1+\frac{c}{a+n}+\frac{d}{(a+n)^2}\right),
\end{aligned}$$

for each $n\in N$, with $n\geq n_0$.

Let $m_0\in N$ be such that $\frac{c-1}{a+n}+\frac{d}{(a+n)(a+n-1)}\in(-1,1]$ and $\frac{c}{a+n}+\frac{d}{(a+n)^2}\in(-1,1]$, for each $n\in N$, with $n\geq m_0$.

We can write that

$$\begin{aligned}
&v_n(a,b,c,d)-v_{n+1}(a,b,c,d)\\
&=b\frac{\varepsilon_n}{1-\varepsilon_n}-(b+1)\varepsilon_n-\ln\left(1+(c-1)\varepsilon_n+d\frac{\varepsilon_n^2}{1-\varepsilon_n}\right)+\ln(1+c\varepsilon_n+d\varepsilon_n^2),
\end{aligned}$$

where $\varepsilon_n:=\frac{1}{a+n}$, for each $n\in N$, with $n\geq n_0$.

Since $\varepsilon_n\in(-1,1)$, $(c-1)\varepsilon_n+d\frac{\varepsilon_n^2}{1-\varepsilon_n}\in(-1,1]$ and $c\varepsilon_n+d\varepsilon_n^2\in(-1,1]$, for each $n\in N$, with $n\geq\max\{n_0,m_0\}$, using the series expansion ([6, pp. 171-179, p. 209]) we obtain

$$\begin{aligned}
&v_n(a,b,c,d)-v_{n+1}(a,b,c,d)\\
&=b\varepsilon_n(1+\varepsilon_n+\varepsilon_n^2+\varepsilon_n^3+\varepsilon_n^4+\cdots)-(b+1)\varepsilon_n\\
&\quad-\varepsilon_n\left(c-1+d\frac{\varepsilon_n}{1-\varepsilon_n}\right)+\frac{1}{2}\varepsilon_n^2\left(c-1+d\frac{\varepsilon_n}{1-\varepsilon_n}\right)^2\\
&\quad-\frac{1}{3}\varepsilon_n^3\left(c-1+d\frac{\varepsilon_n}{1-\varepsilon_n}\right)^3+\frac{1}{4}\varepsilon_n^4\left(c-1+d\frac{\varepsilon_n}{1-\varepsilon_n}\right)^4\\
&\quad-\frac{1}{5}\varepsilon_n^5\left(c-1+d\frac{\varepsilon_n}{1-\varepsilon_n}\right)^5+\cdots\\
&\quad+\varepsilon_n(c+d\varepsilon_n)-\frac{1}{2}\varepsilon_n^2(c+d\varepsilon_n)^2+\frac{1}{3}\varepsilon_n^3(c+d\varepsilon_n)^3\\
&\quad-\frac{1}{4}\varepsilon_n^4(c+d\varepsilon_n)^4+\frac{1}{5}\varepsilon_n^5(c+d\varepsilon_n)^5-\cdots,
\end{aligned}$$

for each $n\in N$, with $n\geq\max\{n_0,m_0\}$. Having in view that

$$c-1+d\frac{\varepsilon_n}{1-\varepsilon_n}=c-1+d\varepsilon_n+d\varepsilon_n^2+d\varepsilon_n^3+d\varepsilon_n^4+\cdots,$$

$$\begin{aligned}
\left(c-1+d\frac{\varepsilon_n}{1-\varepsilon_n}\right)^2&=(c-1)^2+2(c-1)d\varepsilon_n+\left(2(c-1)d+d^2\right)\varepsilon_n^2\\
&\quad+2\left((c-1)d+d^2\right)\varepsilon_n^3+\cdots,
\end{aligned}$$

$$\left(c-1+d\frac{\varepsilon_n}{1-\varepsilon_n}\right)^3=(c-1)^3+3(c-1)^2d\varepsilon_n+3\left((c-1)^2d+(c-1)d^2\right)\varepsilon_n^2+\cdots,$$

$$\left(c-1+d\frac{\varepsilon_n}{1-\varepsilon_n}\right)^4=(c-1)^4+4(c-1)^3d\varepsilon_n+\cdots,$$
$$\left(c-1+d\frac{\varepsilon_n}{1-\varepsilon_n}\right)^5=(c-1)^5+5(c-1)^4d\varepsilon_n+\cdots,$$

it follows that

$$\begin{aligned}
&v_n(a,b,c,d)-v_{n+1}(a,b,c,d)\\
&=\left(b-c+\frac{1}{2}\right)\varepsilon_n^2+\left(b-2d+c^2-c+\frac{1}{3}\right)\varepsilon_n^3\\
&\quad+\left(b-3d+3cd-c^3+\frac{3}{2}c^2-c+\frac{1}{4}\right)\varepsilon_n^4\\
&\quad+\left(b-4d+6cd+2d^2-4c^2d+c^4-2c^3+2c^2-c+\frac{1}{5}\right)\varepsilon_n^5+\cdots,
\end{aligned}$$

for each $n\in N$, with $n\geq\max\{n_0,m_0\}$.

(*i*) Because $b\neq c-\frac{1}{2}$, we can write that

$$\lim_{n\to\infty}n^2\big(v_n(a,b,c,d)-v_{n+1}(a,b,c,d)\big)=b-c+\frac{1}{2}.$$

Now, according to Lemma 1.1, it follows that

$$\lim_{n\to\infty}n\big(v_n(a,b,c,d)-\gamma(a)\big)=b-c+\frac{1}{2}.$$

(*ii*) Because $b=c-\frac{1}{2}$ and $d\neq\frac{1}{2}\left(c^2-\frac{1}{6}\right)$, we can write that

$$\lim_{n\to\infty}n^3\left(v_n\left(a,c-\frac{1}{2},c,d\right)-v_{n+1}\left(a,c-\frac{1}{2},c,d\right)\right)=c^2-2d-\frac{1}{6}.$$

Now, according to Lemma 1.1, it follows that

$$\lim_{n\to\infty}n^2\left(v_n\left(a,c-\frac{1}{2},c,d\right)-\gamma(a)\right)=\frac{1}{2}\left(c^2-\frac{1}{6}\right)-d.$$

(*iii*) Because $b=c-\frac{1}{2}$, $d=\frac{1}{2}\left(c^2-\frac{1}{6}\right)$ and $c\neq0$, $c\neq\pm\frac{\sqrt{2}}{2}$, we can write that

$$\lim_{n\to\infty}n^4\left(v_n\left(a,c-\frac{1}{2},c,\frac{1}{2}\left(c^2-\frac{1}{6}\right)\right)-v_{n+1}\left(a,c-\frac{1}{2},c,\frac{1}{2}\left(c^2-\frac{1}{6}\right)\right)\right)=\frac{c}{2}\left(c^2-\frac{1}{2}\right).$$

Now, according to Lemma 1.1, it follows that

$$\lim_{n\to\infty}n^3\left(v_n\left(a,c-\frac{1}{2},c,\frac{1}{2}\left(c^2-\frac{1}{6}\right)\right)-\gamma(a)\right)=\frac{c}{6}\left(c^2-\frac{1}{2}\right).$$

(*iv*) Because $b=c-\frac{1}{2}$, $d=\frac{1}{2}\left(c^2-\frac{1}{6}\right)$ and $c=0$, we can write that

$$\lim_{n\to\infty}n^5\left(v_n\left(a,-\frac{1}{2},0,-\frac{1}{12}\right)-v_{n+1}\left(a,-\frac{1}{2},0,-\frac{1}{12}\right)\right)=\frac{17}{360}.$$

Now, according to Lemma 1.1, it follows that

$$\lim_{n\to\infty}n^4\left(v_n\left(a,-\frac{1}{2},0,-\frac{1}{12}\right)-\gamma(a)\right)=\frac{17}{1440}.$$

(*v*) Because $b=c-\frac{1}{2}$, $d=\frac{1}{2}\left(c^2-\frac{1}{6}\right)$ and $c=\frac{\sqrt{2}}{2}$, we can write that

$$\lim_{n\to\infty}n^5\left(v_n\left(a,\frac{\sqrt{2}-1}{2},\frac{\sqrt{2}}{2},\frac{1}{6}\right)-v_{n+1}\left(a,\frac{\sqrt{2}-1}{2},\frac{\sqrt{2}}{2},\frac{1}{6}\right)\right)=\frac{1}{180}.$$

Now, according to Lemma 1.1, it follows that

$$\lim_{n\to\infty}n^4\left(v_n\left(a,\frac{\sqrt{2}-1}{2},\frac{\sqrt{2}}{2},\frac{1}{6}\right)-\gamma(a)\right)=\frac{1}{720}.$$

(*vi*) Because $b = c - \frac{1}{2}$, $d = \frac{1}{2}\left(c^2 - \frac{1}{6}\right)$ and $c = -\frac{\sqrt{2}}{2}$, we can write that

$$\lim_{n\to\infty} n^5 \left(v_n\left(a, -\frac{\sqrt{2}+1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{6}\right) - v_{n+1}\left(a, -\frac{\sqrt{2}+1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{6}\right)\right) = \frac{1}{180}.$$

Now, according to Lemma 1.1, it follows that

$$\lim_{n\to\infty} n^4 \left(v_n\left(a, -\frac{\sqrt{2}+1}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{6}\right) - \gamma(a)\right) = \frac{1}{720}.$$

■

Further results regarding Theorem 2.1 can be found in [21].

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CONFERENCE

Spring School on Analysis 2012

What am I if I will not participate?
- Antoine de Saint-Exupery

First announcement
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Dear Colleague,

Following a longstanding tradition, the Faculty of Mathematics and Physics of Charles University in Prague and the Czech Academy of Sciences will organize a Spring School on Variational Analysis V. The School will be held at Paseky nad Jizerou, in a chalet in the Krkonose Mountains, **Apr 22 - 28, 2012**.

The program will consist of series of lectures on

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