***ORIGINAL PAPER***

# INEQUALITIES IN TRIANGLE

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**Abstract.** *In this paper we will demonstrate a new inequality between sine and a function of first degree. Applications of this inequality in triangle are given.*

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## 1. INTRODUCTION

In this paper, we will study the inequalities of type $f(a, b, c, A, B, C, r, s, R) \geq 0$ in a triangle, where $a$, $b$, $c$ are the lengths of sides $AB$, $BC$, $AB$, $A$, $B$, $C$ are the measurements of angles calculated in radians, $r$ is the radius of incircle, $s$ is the semiperimeter and $R$ is the radius of circumcircle.

The following inequality is well known

$$\frac{2x}{\pi} < \sin x < x \tag{1.1}$$

for any $x \in \left(0, \frac{\pi}{2}\right)$, and it is called Jordan's Inequality.

## 2. MAIN RESULTS

In this section, we start with a new inequality, which is a generalization of inequality (1.1).

**Theorem 2.1:** The inequality

$$\sin x \leq \frac{1}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6} \tag{2.1}$$

holds, for any $x \in (0, \pi)$. The equality holds if and only if $x = \frac{\pi}{3}$.

*Proof:* We consider the function $f : (0, \pi) \to R$, defined by:

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$$f(x) = \sin x - \frac{1}{2}x - \frac{\sqrt{3}}{2} + \frac{\pi}{6}$$

Because $f'(x) = \cos x - \frac{1}{2}$, we have the following table:

| $x$ | $0$ | | $\frac{\pi}{3}$ | | $\pi$ |
|---|---|---|---|---|---|
| $f'(x)$ | | $+++$ | $0$ | $---$ | |
| $f(x)$ | | $\nearrow$ | $0$ | $\searrow$ | |

from where, the inequality (2.1) is obtained.

**Corollary 2.1.** In the triangle ABC, we have that

$$\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2} \tag{2.2}$$

and

$$s \le \frac{3R\sqrt{3}}{2} \tag{2.3}$$

*Proof:* Taking (2.1) into account, we have that $\sin A \le \frac{1}{2}A + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$ and the analogous for *B* and *C*. By summing these inequalities, (2.2) follows. Because $\sin A = \frac{a}{2R}$ and $s = \frac{a+b+c}{2}$, from (2.2), it results (2.3).

**Remark 2.1.** The inequality (2.3) is called D.S. Mitrinović's Inequality [2, 4].

**Theorem 2.2.** The inequality

$$\sin x \le \min_{x\in(0,\,\pi)}\left(x,\ \frac{1}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = \begin{cases} x &, \quad x \in \left(0,\ \sqrt{3} - \frac{\pi}{3}\right) \\ \frac{1}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}, & \quad x \in \left[\sqrt{3} - \frac{\pi}{3},\ \pi\right) \end{cases} \tag{2.4}$$

is true.

*Proof:* We have that $\sin x < x$, for any $x \in (0,\ \pi)$, $\sin x \le \frac{1}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$, for any $x \in (0,\ \pi)$, $x < \frac{1}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$ for any $x \in \left(0,\ \sqrt{3} - \frac{\pi}{3}\right)$ and $x > \frac{1}{2}x + \frac{\sqrt{3}}{2} - \frac{\pi}{6}$, for any $x \in \left[\sqrt{3} - \frac{\pi}{3},\ \pi\right)$, from where, the inequality (2.4) results.

**Theorem 2.3.** In any triangle ABC , there are the following inequalities

$$\frac{s\,r}{2R^2}\le\left(\frac{1}{2}A+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)\left(\frac{1}{2}B+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)\left(\frac{1}{2}C+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right) \tag{2.5}$$

and

$$\frac{s^2+r^2+4Rr}{4R^2}\le\sum_{cyclic}\left(\frac{1}{2}A+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)\left(\frac{1}{2}B+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right) \tag{2.6}$$

*Proof:* In this theorem, we use the identities $\mathbf{sin}\,A\,\mathbf{sin}\,B\,\mathbf{sin}\,C=\frac{s\,r}{4R^2}$ and $\sum_{cyclic}\mathbf{sin}\,A\,\mathbf{sin}\,B=\frac{s^2+r^2+4Rr}{4R^2}$ [2, 3, 4]. From inequality (2.1), for $x\in\{A,B,C\}$ we obtain $\mathbf{sin}\,A\le\frac{1}{2}A+\frac{\sqrt{3}}{2}-\frac{\pi}{6}$, $\mathbf{sin}\,B\le\frac{1}{2}B+\frac{\sqrt{3}}{2}-\frac{\pi}{6}$, $\mathbf{sin}\,C\le\frac{1}{2}C+\frac{\sqrt{3}}{2}-\frac{\pi}{6}$.

Multiplying these inequalities, (2.5) follows. Multiplying two by two and taking their sum, we give inequality (2.6).

In [1], we proved the inequality

$$2\left(2-\frac{r}{R}\right)<A^2+B^2+C^2 \tag{2.7}$$

In this paper, we give some refinements of this inequality.

**Theorem 2.4.** In any triangle ABC, we have the inequalities

$$4\left(2-\frac{r}{R}\right)-2\left(\sqrt{3}-\frac{\pi}{3}\right)\pi<A^2+B^2+C^2 \tag{2.8}$$

and if $A,B,C\ge\sqrt{3}-\frac{\pi}{3}$, then

$$4\left(2-\frac{r}{R}\right)-3\left(\sqrt{3}-\frac{\pi}{3}\right)\left(\pi-\sqrt{3}\right)<A^2+B^2+C^2 \tag{2.9}$$

*Proof.* From inequality (2.1), we have

$$\sum_{cyclic}\int_0^A\mathbf{sin}\,x\,dx<\sum_{cyclic}\int_0^A\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)dx,$$

which is equivalent to

$$3-\sum_{cyclic}\mathbf{cos}\,A<\frac{1}{4}\left(A^2+B^2+C^2\right)+\left(\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)\left(A+B+C\right)$$

But $3-\sum_{cyclic}\mathbf{cos}\,A = 2-\frac{r}{R}$ and then, from the inequality above, (2.8) follows.

If $A,\ B,\ C \ge \sqrt{3}-\frac{\pi}{3}$, by using inequality (2.4), we have

$$\sum_{cyclic}\int_0^A \mathbf{sin}\,x dx < \sum_{cyclic}\left(\int_0^{\sqrt{3}-\frac{\pi}{3}} x dx + \int_{\sqrt{3}-\frac{\pi}{3}}^{A}\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)dx\right)$$

which is equivalent to

$$3-\sum_{cyclic}\mathbf{cos}\,A <$$

$$\frac{3}{2}\left(\sqrt{3}-\frac{\pi}{3}\right)^2+\frac{1}{4}\left(A^2+B^2+C^2\right)+\left(\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)\left(A+B+C\right)-\frac{3}{4}\left(\sqrt{3}-\frac{\pi}{3}\right)^2-3\left(\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)\left(\sqrt{3}-\frac{\pi}{3}\right)$$

and by calculus, (2.9) is obtained.

**Theorem 2.5.** In any triangle ABC, where $A,\ B,\ C \ge \sqrt{3}-\frac{\pi}{3}$, we have

$$16\left(\frac{s^2+r^2-2Rr}{R^2}-3\right) < \sum_{cyclic}\left(A^2+2\left(\sqrt{3}-\frac{\pi}{3}\right)A-\left(\sqrt{3}-\frac{\pi}{3}\right)^2\right)\left(B^2+2\left(\sqrt{3}-\frac{\pi}{3}\right)B-\left(\sqrt{3}-\frac{\pi}{3}\right)^2\right) \tag{2.10}$$

*Proof:* From the proof of Theorem 2.4, we have that

$$\int_0^A\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)dx = \frac{1}{4}\left(A^2+2\left(\sqrt{3}-\frac{\pi}{3}\right)A-\left(\sqrt{3}-\frac{\pi}{3}\right)^2\right)$$

and it is well-known that $\sum_{cyclic}\left(1-\mathbf{cos}\,A\right)\left(1-\mathbf{cos}\,B\right) = \frac{s^2+r^2-2Rr}{R^2}-3$.

By using double integrals in (2.4), we have

$$\sum_{cyclic}\int_0^A\int_0^B \mathbf{sin}\,x\,\mathbf{sin}\,y\ dx\ dy < \sum_{cyclic}\int_0^A\int_0^B\left(x,\ \frac{1}{2}x+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)\underset{y\in[0,\ \pi)}{\mathbf{min}}\left(y,\ \frac{1}{2}y+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)dx\ dy,$$

equivalent with

$$\sum_{cyclic}\left(1-\mathbf{cos}\,A\right)\left(1-\mathbf{cos}\,B\right) <$$

$$\sum_{cyclic}\left(\int_0^{\sqrt{3}-\frac{\pi}{3}} x dx+\int_{\sqrt{3}-\frac{\pi}{3}}^{A}\left(\frac{1}{2}x+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)dx\right)\left(\int_0^{\sqrt{3}-\frac{\pi}{3}} y dy+\int_{\sqrt{3}-\frac{\pi}{3}}^{A}\left(\frac{1}{2}y+\frac{\sqrt{3}}{2}-\frac{\pi}{6}\right)dy\right)$$

and taking the remarks above into account, it results (2.10).

**Theorem 2.6.** For any $x \in \left(0, \frac{\pi}{2}\right)$, there is the inequality

$$\cos x + \frac{x\sqrt{3}}{2} \le \frac{1}{2} + \frac{\pi\sqrt{3}}{6}. \tag{2.11}$$

*Proof:* We consider the function $f : \left(0, \frac{\pi}{2}\right) \to R$, defined by

$$f(x) = \frac{1}{2} + \frac{\pi\sqrt{3}}{6} - \cos x - \frac{x\sqrt{3}}{2}.$$

Because $f'(x) = \sin x - \frac{\sqrt{3}}{2}$, we have the following table

| $x$ | $0$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
|---|---|---|---|
| $f'(x)$ | - - - | $0$ | + + + |
| $f(x)$ | $\searrow$ | $0$ | $\nearrow$ |

from where, the inequality (2.11) is obtained.

**Corollary 2.7.** For any $x \in \left(0, \frac{\pi}{2}\right)$, there is the inequality

$$\sin x \le \frac{x\sqrt{3}}{2} - \frac{\pi\sqrt{3}}{12} + \frac{1}{2}. \tag{2.12}$$

*Proof:* In inequality (2.11), if we make the following substitution, $x \to \frac{\pi}{2} - x$, with $x \in \left(0, \frac{\pi}{2}\right)$, then we find the inequality of statement.

**Remark. 2.2.** It is easy to see that $\sin x \le \frac{x\sqrt{3}}{2} - \frac{\pi\sqrt{3}}{12} + \frac{1}{2} \le x$, for $x \ge \frac{6 - \pi\sqrt{3}}{6\left(2 - \sqrt{3}\right)}$, representing another refinement for the part two of Jordan's inequality.

**Corollary 2.8.** In the acute triangle ABC, we have that

$$\cos A + \cos B + \cos C \le \frac{3}{2} \tag{2.13}$$

and

$$R \ge 2r \tag{2.14}$$

*Proof:* Taking into account, from (2.11), that $\cos A+\frac{A\sqrt{3}}{2}\le\frac{1}{2}+\frac{\pi\sqrt{3}}{6}$ and the analogous for *B* and *C*. By summing these inequalities, (2.13) follows. Because, we have the identity

$$\cos A+\cos B+\cos C=1+\frac{r}{R},$$

from [3,4], it is easy to see that we obtain (2.14), which is due to Euler.

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